Abstract algebra

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Chapter 2

Infinite-dimensional vector spaces

2.1 Real functions as infinite-dimensional vectors

2.1.1 Real functions are vectors

The real function space is a vector space because it is linear in multiplication and addition.

$g(x) = cf(x)$

$h(x) = f(x) + k(x)$

2.2 Endomorphisms of infinite-dimensional vector spaces

2.2.1 Endomorphisms on real functions

We start with our vector $f(x)$.

$h(x) = f(x)g(x)$

The equivalent of the identity matrix is where $g(x) = 1$.

These are similar to endomorphisms where all off diagonal elements are 0.

Differentiation

$h(x) = \frac{\delta}{\delta x} f(x)$

Integration

$h(x) = \int_{-\infty}^{x} f(z)dz$

2.2.2 Examples of linear operators on real functions

For a function v we can define operators Ov.

Here we consider some examples and their properties.

Real multiplication

$Rv = rf(x)$

This operator is hermitian. This is equivalent to a finite operator of the form rI.

Multiplication by underlying real number

$Xv = xf(x)$

This operator is hermitian. This is equivalent to a finite operator of the form $M_{ii} = i$ and $M_{ij} = 0$.

Differentiation

$Dv = \frac{\delta}{\delta x} f(x)$

While this operator is not hermitian, the following is:

$-iDv = \frac{\delta}{\delta x}[-if(x)]$

2.3 Eigenvalues and eigenvectors of infinite-dimensional vectors

2.3.1 Spectral theorem for infinite-dimensional vector spaces

2.4 Forms on infinite-dimensional vector spaces

2.4.1 Forms on real functions

A form takes two vectors and produces a scalar.

Integration as a form

We can use integration to get a bilinear form.

$\int f(x)g(x)dx$

If we instead want a sesquilinear form we can instead use:

$\int f(\bar{x})g(x)dx$

2.4.2 Functionals

Functionals map functions to scalars. They are the 1-forms of infinite-dimensional vector spaces.

If we have a function f, we can write functional $J[f]$.

More

We can define neighbourhoods around a function f. For example, taking y to be f with infintesimal changes. to each of the values.

The difference between the functional at both points is

$\delta J = J[y] - J[f]$

Extrema

If

$\delta J = J[y] - J[f]$

is the same sign for all y around f, then J has an extremum at f.

Functional derivatives

2.4.3 Hilbert space

A complete space with an inner product. That is, a Banach space where the norm is derived from an inner product.

2.5 Calculus of variations

2.5.1 Calculus of variations

2.5.2 Functional integration

Integrate over possible functions?

2.6 Sort

2.6.1 Banach space

A complete normed vector space

2.6.2 Wave functions

For a vector in hermitian basis, for each eigenvector we have component. wave function is function on ith component.

2.7 Other

2.7.1 Dirac delta

Kronecker delta

The function is: δ_{ij}

If $i = j$ this is 1. Otherwise it is 0.

We introduced this in linear algebra.

Dirac delta

The Dirac delta replaces the Kronecker delta for continuous functions.

That is, we want:

- $\delta(x \neq 0) = 0$
- $\delta(0) = +\infty$
- $\int_{-\infty}^{\infty} \delta(x) dx = 1$

Part II

Group theory of countable groups

Chapter 3

Magmas (groupoids), semigroups, monoids and groups

3.1 Defining groups

3.1.1 Magma

A magma, or groupoid, is a set with a single binary operation.

These can be defined as an ordered pair $(s, \odot)$ where s is the set, and $\odot$ is the binary operation.

If a and b are in s, then $a \odot b$ is also in s.

The following are magmas:

- Natural numbers and addition
- $n \times n$ matrices with determinants other than 0
- Natural numbers above 0 and addition
- Integers and addition
- Rational numbers and division
- $\{-1, 1\}$ and multiplication

The following are not magmas:

- Natural numbers up to 10 and addition

3.1.2 Semigroup

A semigroup is a magma whose binary operation is associative.

The following are semigroups:

- Natural numbers and addition
- $n \times n$ matrices with determinants other than 0
- Natural numbers above 0 and addition
- Integers and addition
- $\{-1, 1\}$ and multiplication

The following are not semigroups:

- Rational numbers and division
- Natural numbers up to 10 and addition

3.1.3 Monoid

A monoid is a semigroup with an identity element.

The following are monoids:

- Natural numbers and addition
- $n \times n$ matrices with determinants other than 0
- Integers and addition
- $\{-1, 1\}$ and multiplication

The following are not monoids:

- Natural numbers above 0 and addition
- Rational numbers and division
- Natural numbers up to 10 and addition

3.1.4 Group

A group is a monoid where there is an inverse operation for the binary operation.

The following are groups:

- Integers and addition
- $n \times n$ matrices with determinants other than 0
- $\{-1, 1\}$ and multiplication

The following are not groups:

- Natural numbers above 0 and addition
- Rational numbers and division
- Natural numbers and addition
- Natural numbers up to 10 and addition

Chapter 4

Homomorphisms and isomorphisms to Z with addition

4.1 Introduction

4.1.1 Homomorphism

Homomorphisms are functions which preserve the relationships between members of a set, and specified functions.

That is, if:

$a \odot b = c$

Then $f(x)$ is morphism if:

$f(a) \odot f(b) = f(a \odot b)$

Here we discuss morphisms in the context of groups, but we can define morphisms for sets with more than one function, for example with addition and multiplication.

Morphisms are also known as homomorphisms.

The following are morphisms of the additive group of integers.

Where we refer to c, $c \neq 0 \in \mathbb{I}$.

- $f(x) = 0$
- $f(x) = x$
- $f(x) = cx$

- Converting natural numbers to integers

The following are not morphisms

- $f(x) = x + 1$

4.1.2 Isomorphism

An isomorphism is a morphism which has an inverse.

This means the function is bijective.

The following are isomorphisms:

- $f(x) = x$
- $f(x) = cx$
- Converting natural numbers to integers

The following are not isomorphisms

- $f(x) = 0$
- $f(x) = x + 1$

4.2 The infinite cyclic group (Z)

4.2.1 Infinite cyclic groups are isomorphic to the additive group of integers

More generally, any infinite cyclic group is isomorphic to the additive group of integers.

Chapter 5

Finite cyclic groups (C_n aka Z_n aka Z/nZ)

5.1 Finite groups

5.1.1 Group order

For finite groups, each element e has:

$e^n = I$

For some $n \in \mathbb{N}$

Where I is the identity element.

The order of the group is the smallest value of n such that that holds for all elements.

For example in the multiplicative group $G = \{-1, 1\}$ the order is 2.

Or:

$|G| = 2$

Additionally

$|-1| = 2$

$|1| = 1$

5.1.2 Finite groups

Consider the set of natural numbers and addition modulo 4. This forms a group containing:

$\{0, 1, 2, 3\}$

This can be written as Z_4 or more generally as Z_n, or Z/nZ.

5.1.3 The trivial group

The trivial group is the group with just the identity member I.

5.1.4 The finite cyclic groups (C_n or Z_n)

Consider the multiplicative group of $< i >$.

This contains $\{1, -1, i, -i\}$.

This is also automorphic to the natural number and modulo addition group above.

We can define finite cyclic groups of size n using the generating element $z^{\frac{1}{n}}$. This is isomorphic to the general cyclic group C_n, and to Z/nZ.

Chapter 6

Symmetric groups (S_n)

6.1 Creating groups

6.1.1 Permutations and the symmetric group

A permutation is defined as a bijection from a set to itself.

For a set of size n, the number of permutations is $n!$. This is because there are n possibilities for the first item, $n-1$ for the second and so on.

The symmetric group

The set of all permutations forms a group, the symmetric group. This forms a group because:

- There is an identity element
- Each combination of permutations is also in the group.
- Each permutation has an inverse in the group.

Permutation groups

A subgroup of the symmetric group is called a permutation group.

6.2 Abelian groups

6.2.1 Abelian groups

A commutative group, that is where $a \odot b = b \odot a$.

The following are abelian groups:

- Integers and addition

- $\{-1, 1\}$ and multiplication

The following are not abelian groups:

- $n \times n$ matrices with determinants other than 0

6.2.2 The group commutator

The group commutator is:

$[a, b] = a^{-1}b^{-1}ab$

If the group is abelian then $[a, b] = 0$. The group commutator is a measure of how non-abelian the group is.

This has the following properties:

- Alternativity: $[A, A] = I$

6.3 Direct product

6.3.1 The direct product of groups

If we have two groups G and H we can form new group $G \times H$.

For every $g \in G$ and $h \in H$ there is $(g, h) \in G \times H$.

The binary operation we have is:

$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1, h_2)$

Chapter 7

Subgroups, generating sets, cosets, normal subgroups, Cayley's theorem and Lagrange's theorem

7.1 Introduction

7.1.1 Introduction

7.1.2 Subgroups

A subgroup of a group is a subset of a group, which also forms a group with the same element.

For example all even numbers are a subgroup of the addition group of integers.

7.2 Generating cyclic groups

7.2.1 Generating cyclic groups

We can generate a group with a single element, it is a cyclic group.

For example, we can define a group $G =< 1 >$ which gives us the additive group of integers.

7.2.2 Generating sets

We can define a group through a generating set and an operation.

And define the group as $G =< S >$

7.3 Normal subgroups

7.3.1 Cosets and normal subgroups

A coset is defined between a group and a subgroup of the group.

For a group G, and its subgroup H:

- The left coset is $\{gH\}$
- The right coset is $\{Hg\}$

For $\forall g \in G$.

For abellian groups, the left and right cosets are the same.

The left and right cosets can also be the same, even if the group G is not abelian.

Normal subgroups

If the left and right cosets are the same then H is a normal subgroup.

Cosets divide a group.

Consider two left cosets, aH and bH, with a common element.

This means that $ah_i = bh_j$.

We can use this to get:

$a = bh_j h_i^{-1}$

$b = ah_i h_j^{-1}$

We know that:

$ah \in aH$

$bh \in bH$

So:

$bh_j h_i^{-1} h \in aH$

$ah_i h_j^{-1} h \in bH$

And so:

$bH \subset aH$

$aH \subset bH$

Therefore:

$aH = bH$

Example 1

Consider the group $\{-1, 1\}, \times$

For the subgroup $\{1\}, \times$, the left coset is $\{gH\} = \{1, -1\}$.

The right coset is the same.

Example 2

Consider the group of integers and addition: $(Z, +)$

For subgroup $(mZ, +)$, the left and right cosets are the same because the group is abelian.

The coset of the subgroup is the subgroup multiplied by each element in G.

This is mZ, $mZ + 1$, $mZ + 2$ and so on.

Once we reach $mZ + m$ this has looped, and is already a coset, so we only need the sets upto $mZ + m - 1$.

7.3.2 Quotient groups

We have a group G and a normal subgroup N.

We define a quotient group from this as G/N. This is the set of cosets from N.

7.3.3 Group extension

This defines a group G from a normal subgroup N and a quotient group Q.

7.4 Theorems

7.4.1 Cayley's theorem

Cayley's theorem states that every group G is isomorphic to a subgroup of the symmetric group acting on G.

Multiplication by a member of G is a bijective function, as for each g there is also a g^{-1}.

This means that multiplication of each member of G is a permutation, and so is a subset of the symmetric group on G.

7.4.2 Lagrange's theorem

Lagrange's theorem states that for any finite group G, the order of every subgroup is a divisor of the order of G.

Consider subset H. We know that all cosets are disjoint, and that the union of all cosets is G.

As cosets are the same size, we know that:

$|G| = m|H|$, where m is the number of cosets.

This means that if a group has order 10, a subgroup must have order 1, 2 5 or 10.

Chapter 8

Monomorphisms, epimorphisms

8.1 Morphisms

8.1.1 Monomorphism

A morphism which is injective. That is:

$f(a) = f(b) \to a = b$

The following are monomorphisms:

- $f(x) = x$
- $f(x) = cx$
- Converting natural numbers to integers

The following are not monomorphisms:

- $f(x) = 0$
- $f(x) = x + 1$

Chapter 9

Group theory: Linear

9.1 Introduction

9.1.1 Introduction

9.2 Group action

9.2.1 Group action

We have a group G and a set S.

We have a function $g.s$ which maps onto S such that:

- $I.s = s$
- $(gh).s = g(h.s)$

Chapter 10

Endomorphisms and automorphisms

10.1 Introduction

10.1.1 Endomorphism

An endomorphism is one where the domain and codomain are the same.

The following are endomorphisms:

- $f(x) = 0$
- $f(x) = x$
- $f(x) = cx$

The following are not endomorphisms

- Converting natural numbers to integers
- $f(x) = x + 1$

10.1.2 Automorphism

An endomorphism which is also an isomorphism

The following are automorphisms:

- $f(x) = x$
- $f(x) = cx$

The following are not automorphisms

- $f(x) = 0$

- $f(x) = x + 1$
- Converting natural numbers to integers

Part III

Group theory of uncountable groups

Chapter 11

Uncountable groups

11.1 Uncountable groups

11.1.1 The circle group T

The circle group, T, includes all complex numbers of magnitude 1.

Chapter 12

Lie algebra

12.1 Lie groups

12.1.1 Lie groups

12.2 Lie algebra

12.2.1 Lie algebra

Lie groups have symmetries. We can consider only the infintesimal symmetries.

For example the unit circle has many symmetries, but we can consider only those which rotate infintesimally.

Example

Take a continous group, such as $U(1)$. Its Lie algebra is all matrices such that their exponential is in the Lie group.

$\mathfrak{u}(1) = \{X \in \mathbb{C}^{1\times 1} | e^{tX} \in U(1) \forall t \in \mathbb{R}\}$

This is satisfied by the matrices where $M = -M^*$. Note that this means the diagonals are all 0.

Scale of specific Lie algebra matrices doesn't matter

Because of t.

Commutation of Lie group algebra

Consider two members of the Lie algebra: A and B. The commutator is:

A.

The corresponding Lie group member is:

$e^{t(A+B)} = e^{tA}e^{tB}$

While the Lie group multiplication may not commute, the corresponding addition of the Lie algebra does.

12.2.2 The Lie bracket

We can define the Lie bracket from the ring commutator.

We use the Lie bracket, rather than multiplication, as the operator over a field homomorphism.

$[A, B]$

This generates another element in the algebra.

This satisifies:

- Bilinearity: $[xA + yB, C] = x[A, C] + y[B, C]$
- Alternativity: $[A, A] = 0$
- Jacobi identity: $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$
- Anticommutivity: $[A, B] = -[B, A]$

One option for the Lie bracket is the ring commutor. So that:

$[A, B] = AB - BA$

12.2.3 Commutation of Lie groups

We can measure commutation of Lie groups using:

$ABA^{-1}B^{-1}$

If the group commutes then:

$ABA^{-1}B^{-1} = BA^{-1}B^{-1} = I$

Commutation of Lie algebra: COMPLETE THIS

This corresponds to $[A, B] = AB - BA$ in the underlying lie algebra, if we expand.

$A = e^{ta}$

$B = e^{tb}$

$ABA^{-1}B^{-1} = e$

12.3 Lie algebra of specific Lie groups

12.3.1 Lie algebra of $O(n)$

$O(n)$ forms a Lie group

Lie algebra of $O(n)$

The Lie algebra of $O(n)$ is defined as:

$\mathfrak{o}(n) = \{X \in \mathbb{R}^{n\times n} | e^{tX} \in O(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-symmetric matrices where $M = -M^T$. Note that this means the diagonals are all 0.

12.3.2 Lie algebra of $U(n)$

$U(n)$ forms a Lie group

Lie algebra of $U(n)$

The Lie algebra of $U(n)$ is defined as:

$\mathfrak{u}(n) = \{X \in \mathbb{C}^{n\times n} | e^{tX} \in U(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-Hermitian matrices where $M = -M^*$. Note that this means the diagonals are all 0 or pure imaginary.

12.3.3 Lie algebra of $SO(n)$

$SO(n)$ forms a Lie group

Lie algebra of $SO(n)$

The Lie algebra of $SO(n)$ is defined as:

$\mathfrak{so}(n) = \{X \in \mathbb{R}^{n\times n} | e^{tX} \in SO(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-symmetric matrices where $M = -M^T$. Note that this means the diagonals are all 0.

12.3.4 Lie algebra of $SU(n)$

$SU(n)$ forms a Lie group

Lie algebra of $SU(n)$

The Lie algebra of $SU(n)$ is defined as:

$\mathfrak{su}(n) = \{X \in \mathbb{C}^{n\times n} | e^{tX} \in SU(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-Hermitian matrices where $M = -M^*$ and the trace is 0. Note that this means the diagonals are all 0 or pure imaginary.

12.4 Hypercomplex numbers

12.4.1 Hypercomplex numbers

12.4.2 Quaternions

12.4.3 Clifford algebra

12.5 Sort

12.5.1 Projective line in the field

Part IV

Rings and fields

Chapter 13

Rings

13.1 Introduction

13.1.1 Rings

Consider an abelian group $(S, +)$.

A ring takes this and adds a multiplicative function which satisfies the distributive property.

Groups have an identity element for their function. Rings must have identity elements for both their functions.

The multiplicative function does not have to be a bijection. For example the set of integers, addition and multiplication form a ring.

13.1.2 Rngs

A rng is a ring without the multiplicative identity (hense no 'i').

13.2 Commutation

13.2.1 Commutative rings

The multiplication operation commutes.

13.2.2 Commutator

$[a, b] = ab - ba$

13.2.3 The Jacobi identity

13.3 Examples of rings

13.3.1 Zero (trivial) ring

The trivial ring is a ring with just one element.

0 with addition and multiplication work.

13.3.2 Integer rings

The integers with addition and multiplication form a ring.

13.3.3 Integer mod n rings

The integers mod n with addition and multiplication form a group.

Examples

The integers $\{1, 2, 3\}$ form a ring.

13.4 Properties of rings

13.4.1 Characteristic of a ring

The characteristic of a ring is the number of times the multiplicative identity must be added to get the additive identity.

If this never happens, the characteristic is 0.

Example

The integer mod 2 ring, the characteristic is 2.

13.5 Division

13.5.1 Division rings

A divison ring is a ring where every non-zero element has a multiplicative inverse.

Example

The rational numbers are a divison ring.

Relationship between divison rings and fields

Fields (not yet introduced) are different from division rings only in that multiplication for a field must be commutative.

13.5.2 Units

A unit is an element of a ring which has a multiplicative inverse.

Examples

The ring of integers with addition and multiplication, only -1 and 1 are units, as both have multiplicative inverses in the ring.

13.6 Subrings

13.6.1 Subrings

A subring is a subset of the ring, where the addition and multiplication operations on the subring result in elements also in the subring.

Example

The even numbers are a subring of the integers.

13.6.2 Ideals

An ideal is a subring where the multiplication of any element of the ideal with any element of the ring is also in the ideal.

Examples

Even numbers are an ideal of the integers.

Odd numbers are not an ideal. For example 1 is in the ideal, but multiplied by 2 gives 2, which is not in the ideal.

Chapter 14

Fields

14.1 Fields

14.1.1 Fields

A field is a ring where the multiplication function has an inverse.

The integers, addition and multiplication form a ring, but not a group.

The rational numbers (except 0), addition and multiplication form a field (and a ring).

The real numbers and complex numbers also form fields.

14.2 Finite (Galois) fields

14.2.1 Introduction

Finite number of elements.

Integers mod p field

Where p is prime.

Characteristic of a field

14.3 Algebra on a field

14.3.1 Bilinear maps

A bilinear map (or function) is a map from two inputs to an output which preserves addition and scalar multiplication. This is in contrast to a linear map, which only has one input.

In addition, the function is linear in both arguments.

That is if function f is bilinear then:

$X = aM + bN$

$Y = cO + dP$

$f(X, Y) = f(aM + bN, cO + dP)$

$f(X, Y) = f(aM, cO + dP) + f(bN, cO + dP)$

$f(X, Y) = f(aM, cO) + f(aM, dP) + f(bN, cO) + f(bN, dP)$

$f(X, Y) = acf(M, O) + adf(M, P) + bcf(N, O) + bdf(N, P)$

Note that:

$f(X, Y) = f(X + 0, Y)$

$f(X, Y) = f(X, Y) + f(0, Y)$

$(0, Y) = 0$

That is, if any input is 0 in an additative sense, the value of the map must be zero.

14.3.2 Algebra on a field

Part V

Abstract linear algebra

Chapter 15

Vector spaces

15.1 Vector spaces

15.1.1 Vector spaces

A vector space is a group with additional structure.

The operation for each element is shown as addition. So we can say:

$\forall u, v \in V[u + v \in V]$

To this we add scalars, from a field F. We write this as multiplication.

$\forall f \in F \forall v \in V[fv \in V]$

Subspace

A subspace is a subset of V which still acts as a vector space. In practice, this means fewer dimensions.

15.1.2 Span

Span function

We can take a subset S of V. We can then make linear combinations of these elements.

This is called the linear span - $span(S)$.

15.1.3 Linear dependence

A collection of vectors in a vector space are linearly dependent if there exist values for α (other than all being 0) such that:

$\sum_i \alpha_i v_i = 0$.

If no such values for α exist we say the vectors are linearly independent.

15.1.4 Basis vectors

Basis

We can write vectors as combinations of other vectors.

$v = \sum_i \alpha_i v_i$

A subset which spans the vector space, and which is also linearly independent, is a basis of the vector space.

For an arbitrary vector of size n, we cannot use less than n elementary vectors. We could use more, but these would be redundant.

If we use n elementary vectors, there is a unique solution of weights of elementary vectors.

If we use more than n elementary vectors, there will be linear dependence, and so there will not be a unique solution.

15.1.5 Dimension function

For a basis S, the the dimension of the vector space is $|S|$.

$\dim(V) = |S|$

$S \subset V$

Finite and infinite vector spaces

If $\dim(V)$ is finite, then we say the vector space is finite.

Otherwise, we say the vector space is infinite.

15.2 Points, lines and planes

15.2.1 Points, lines and planes

$(1, 0)$ is point, $(x, 2x + 1)$ is a line $(1, x, y)$ is a plane

15.2.2 Parallel lines and planes

Parallel lines

If we have two lines:

Parallel planes

Chapter 16

Linear endomorphisms

16.1 Endomorphisms of vector spaces

16.1.1 Endomorphisms

An endomorphism maps a vector space onto itself.

$end(V) = \hom(V, V)$

16.1.2 Endomorphisms form a vector space

An endomorphism maps a vector space onto itself.

$end(V) = \hom(V, V)$

Need to show that endomorphism is a vector space

Essentially

$v \in V$

$f \in F$

$av = f$

$bv = g$

$(a \oplus b)v = f + g$

$(a \oplus b)v = av + bv$

so there is some operation we can do on two members of endo

linear in addition. That is, if we have two dual "things", we can define the addition of functions as the operation which results int he outputs being added.

what about linear in scalar? same approach.

Well we define

$c \odot a) = cav$

There is a unique endomorphism which results in two other endomorphisms being added together. define this as addition

16.1.3 Dimension of endomorphisms

$\dim(end(V)) = (\dim V)^2$

16.1.4 Basis of endomorphisms

16.1.5 Projections

A projection is a linear map which if applied again returns the original result.

A projection can drop a dimension for example.

16.1.6 Kernels and images

The kernel of a linear operator is the set of vectors such that:

$Mv = 0$

The kernel is also called the nullspace.

This can be shown as $\ker(M)$

The image of a linear operator is the set of vectors w such that:

$Mv = w$.

This can be shown as $\Im(M)$

We also know that:

$span(M) = \ker(M) + \Im(M)$

16.2 Representing endomorphisms with matrices

16.2.1 Matrix representation

Representing linear maps as matrices

We previously discsussed morphisms on vector spaces. We can write these as matrices.

Matrices represents transformations of vector spaces

Representing vectors as matrices

We can represent vectors as row or column matrices.

$v = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix}$

$v = \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_m \end{bmatrix}$

16.3 Automorphisms of vector spaces

16.3.1 Basis of an endomorphism

16.3.2 Changing the basis

For any two bases, there is a unique linear mapping from of the element vectors to the other.

16.4 The linear groups

16.4.1 General linear groups $GL(n, F)$

The general linear group, $GL(n, F)$, contains all $n \odot n$ invertible matrices M over field F.

The binary operation is multiplication.

16.4.2 Endomorphisms as group actions

We can view each member of the group g as a homomorphim on s.

Where s is a vector space V, the representation on each group member is an invertible square matrix.

If the set we use is the vector space V, then we can represent each group element with a square matrix acting on V.

Faithful means $a \neq b$ holds for repesentation too.

Representation theory. groups defined by $ab = c$. if we can match each eleemnt to amatrix where this holds we have represented the matrix.

16.4.3 Representing finite groups

Finite groups can all be represented with square matrices.

16.4.4 Representing compact groups

Chapter 17

Linear forms

17.1 Linear forms

17.1.1 Linear forms

A linear form is a linear map from a vector space to a scalar from the vector space's underlying field.

$\hom(V, F)$

Matrix operators

Linear forms can be represented as matrix operators.

$v^T M = f$

Where M has only one column.

Stuff

$f(M) = f(v)$

We introduce e_i, the element vector. This is 0 for all entries except for i where it is 1. Any vector can be shown as a sum of these vectors multiplied by a scalar.

$f(M) = f(\sum_{i=1}^m a_i e_i)$

$f(M) = \sum_{i=1}^m f(a_i e_i)$

$f(M) = \sum_{i=1}^m a_i f(e_i)$

$f(M) = \sum_{i=1}^m a_i f(e_i)$

Orthonormal basis

$f(M) = \sum_{i=1}^{m} a_i$

17.1.2 Dual space

The dual space V^* of vector space V is the set of all linear forms, $\hom(V, F)$.

The dual space is itself a vector space

$v \in V$

$f \in F$

$av = f$

$bv = g$

$(a \oplus b)v = f + g$

$(a \oplus b)v = av + bv$

So there is some operation we can do on two members of dual space

Linear in addition. That is, if we have two dual "things", we can define the addition of functions as the operation which results int he outputs being added.

what about linear in scalar? same approach.

Well we define

$(c \odot a) = cav$

The dual space has the same dimension as the underlying vector space

17.1.3 The dual space forms a vector space

The dual space forms a vector space. We can define addition and scalar multiplication on members of the dual space.

The dimension of the dual space is the same as the underlying space.

We have defined the dual space. A vector in dual space will have also have components and a basis.

$\mathbf{w} = \sum_i w_i f^j$

So how we describe the components will depend on the choice of basis.

We choose the dual basis, the basis for V^* as:

$\mathbf{e}_i \mathbf{f}^j = \delta_i^j$

If the basis changes, so does the dual basis.

We write the dual basis as e^j

17.2 Bilinear forms

17.2.1 Bilinear forms

A bilinear form takes two vectors and produces a scalar from the underyling field.

This is in contrast to a linear form, which only has one input.

In addition, the function is linear in both arguments.

$\phi(au+x,bv+y)=\phi(au,bv)+\phi(au,y)+\phi(x,bv)+\phi(x,y)$

$\phi(au+x,bv+y)=ab\phi(u,v)+a\phi(u,y)+b\phi(x,v)+\phi(x,y)$

Representing bilinear forms

They can be represented as:

$\phi(u,v)=v^TMu$

$f(M)=f([v_1,v_2])$

We introduce e_i, the element vector. This is 0 for all entries except for i where it is 1. Any vector can be shown as a sum of these vectors multiplied by a scalar.

$f(M)=f([\sum_{i=1}^m a_{1i}e_i,\sum_{i=1}^m a_{2i}e_i])$

$f(M)=\sum_{k=1}^m f([a_{1k}e_k,\sum_{i=1}^m a_{2i}e_i])$

$f(M)=\sum_{k=1}^m\sum_{i=1}^m f([a_{1k}e_k,a_{2i}e_i])$

Because this in linear in scalars:

$f(M)=\sum_{k=1}^m\sum_{i=1}^m a_{1k}a_{2i}f([e_k,e_i])$

$f(M)=\sum_{k=1}^m\sum_{i=1}^m a_{1k}a_{2i}e_k^TMe_i$

Orthonormal basis and $M=I$

$f(M)=\sum_{k=1}^m\sum_{i=1}^m a_{1k}a_{2i}e_k^Te_i$

$f(M)=\sum_{k=1}^m\sum_{i=1}^m a_{1k}a_{2i}\delta_i^k$

$f(M)=\sum_{i=1}^m a_{1i}a_{2i}$

17.2.2 The dot product

$v^TMu=f$

If the operator is I then we have the dot product.

v^Tu

17.2.3 Orthogonal vectors

Given a metric M, two vectors v and u are orthogonal if:

$v^T M u = 0$

For example if we have the metric $M = I$, then two vectors are orthogonal if:

$v^T u = 0$

17.2.4 Metric-preserving transformations and isometry groups

If we have a bilinear form we can write the form as:

$u^T M v$

After a transformation P to the vectors it is:

$(Pu)^T M (Pv)$

$u^T P^T M P v$

So the value of the metric will be unaffected if:

$u^T P^T M P v = u^T M v$

$P^T M P = M$

Equivalent metrics

Different metrics can produce the same group. For example multiplying the metric by a constant.

$P^T M P = M$

17.2.5 Orthogonal groups $O(n, F)$

Recap: Metric-preserving transformations

The bilinear form is:

$u^T M v$

The transformations which preserve this are:

$P^T M P = M$

The orthogonal group

If the metic is $M = I$ then the condition is:

$P^T P = I$

$P^T = P^{-1}$

These form the orthogonal group.

We use O instead of P:

$O^T = O^{-1}$

Rotations and reflections

The orthogonal group is the rotations and reflections.

Parameters of the orthogonal group

The orthogonal group depends on the dimension of the vector space, and the underlying field. So we can have:

- $O(n, R)$; and
- $O(n, C)$.

We generally refer only to the reals

$O(n)$ means $O(n, R)$.

The generally refer to the reals only.

17.2.6 Indefinite (pseduo) and split orthognal groups $O(n, m, F)$

Recap: Metric-preserving transformations

The bilinear form is:

$u^T M v$

The transformations which preserve this are:

$P^T M P = M$

The metric

If the metric is:

$$M = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then we have the indefinite orthogonal group $O(3, 1)$

The split orthogonal group

Where $n = m$ we have the split orthogonal group.

$O(n, n, F)$

Signatures

17.2.7 The Lorentz group

The Lorentz group is the $O(1,3)$ group.

Symmetries of the Lorentz group

We can do the usual 3 rotations, however there are additional 3 symmetries, making the Lorzentz group 6-dimensional.

These are the Lorentz boosts.

A symmetry has:

$t'^2 - x'2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2$

We consider the case where we just boost on x, so $y = y'$ and $z = z'$.

$t'^2 - x'2 = t^2 - x^2$

Or with c:

$c^2 t'^2 - x'2 = ct^2 - x^2$

New

$s^2 = t^2 - x^2 - y^z - z^2$

$s'^2 = t'^2 - x'^2 - y'^2 - z'^2$

$ds^2 = s'^2 - s^2$

$ds^2 = (t'^2 - x'^2 - y'^2 - z'^2) - (t^2 - x^2 - y^z - z^2)$

$ds^2 = (t'^2 - t^2) - (x'^2 - x^2) - (y'^2 - y^2) - (z'^2 - z^2)$

$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$

boost: $s^2 = c^2 t^2 - x^2 - y^z - z^2$

we want new t and x where distance is same $c^2 t'^2 - x'^2 - y^z - z^2 = c^2 t^2 - x^2 - y^z - z^2$ $c^2 t'^2 - x'^2 = c^2 t^2 - x^2$

We know that both transformations are linear [WHY??], therefore $x' = Ax + Bt$ $t' = Cx + Dt$

we transform to $x' = 0$. so $Ax + Bt = 0$

We define $v = \frac{x}{t}$

So: $x = vt$

We can plug these in: $Avt + Bt = 0$ $Av + B = 0$ $\frac{A}{B} = -v$

17.3 Sesquilinear forms

17.3.1 Sesquilinear forms

Bilinear form recap

A bilinear form takes two vectors and produces a scalar from the underyling field.

The function is linear in addition in both arguments.

$\phi(au+x, bv+y) = \phi(au, bv) + \phi(au, y) + \phi(x, bv) + \phi(x, y)$

The function is also linear in multiplication in both arguments.

$\phi(au+x, bv+y) = ab\phi(u, v) + a\phi(u, y) + b\phi(x, v) + \phi(x, y)$

They can be represented as:

$\phi(u, v) = v^T Mu$

Sesquilinear forms

Like bilinear forms, sesquilinear are linear in addition:

$\phi(au+x, bv+y) = \phi(au, bv) + \phi(au, y) + \phi(x, bv) + \phi(x, y)$

Sesqulinear forms however are only multiplictively linear in the second argument.

$\phi(au+x, bv+y) = b\phi(au, v) + \phi(au, y) + b\phi(x, v) + \phi(x, y)$

In the first argument they are "twisted"

$\phi(au+x, bv+y) = \bar{a}b\phi(u, v) + \bar{a}\phi(u, y) + b\phi(x, v) + \phi(x, y)$

The real field

For the real field, $\bar{b} = b$ and so the sesqulinear form is the same as the bilinear form.

Representing sesquilinear forms

We can show the sesquilinear form as $v^* Mu$

Stuff

$f(M) = f([v_1, v_2])$

We introduce e_i, the element vector. This is 0 for all entries except for i where it is 1. Any vector can be shown as a sum of these vectors multiplied by a scalar.

$f(M) = f([\sum_{i=1}^m a_{1i}e_i, \sum_{i=1}^m a_{2i}e_i])$

$f(M) = \sum_{k=1}^m f([a_{1k}e_k, \sum_{i=1}^m a_{2i}e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m f([a_{1k}e_k, a_{2i}e_i])$

Because this in linear in scalars:

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} f([e_k, e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} e_k^* M e_i$

Orthonormal basis and $M = I$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} e_k^* M e_i$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} e_k^* e_i$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} \delta_i^k$

$f(M) = \sum_{i=1}^m a_{1i}^* a_{2i}$

17.3.2 Unitary groups $U(n, F)$

Metric preserving transformations for sesquilinear forms

For bilinear forms, the transformations which preserved metrics were:

$P^T = P^{-1}$

For sesquilinear they are different:

$u^* M v$

$(Pu)^* M (Pv)$

$u^* P^* M P v$

So we want the matrices where:

$P^* M P = M$

The unitary group

The unitary group is where $M = I$

$P^* P = I$

$P^* = P^{-1}$

We refer to these using U instead of P.

$U^* = U^{-1}$

Parameters of the unitary group

The unitary group depends on the dimension of the vector space, and the underlying field. So we can have:

- $U(n, R)$; and

- $U(n, C)$.

We generally refer only to the complex

For the $U(n, R)$ we have:

$U^* = U^{-1}$

$U^T = U^{-1}$

This is the condition for the orthogonal group, and so we would instead write $O(n)$.

As a result, $U(n)$ refers to $U(n, C)$.

$U(1)$**: The circle group**

17.4 Inner products

17.4.1 Symmetric matrices

17.4.2 Hermitian (self-adjoint) matrices

A matrix where $M = M^*$

For matrices over the real numbers, these are the same as symmetric matrices.

Sesqulinear forms on Hermitian matrices

$\phi(u, v) = u^* M v$

$(u^* M v)^* = v^* M^* u = v^* M u$

$\phi(u, v) = \overline{\phi(v, u)}$

The forms on the same vector are always real

$(v^* M v)^* = v^* M^* v = v^* M v$

So we have:

$(v^* M v)^* = v^* M v$

Which is only satisfied for reals.

If A and B are Hermitian

If A and B are Hermitian, AB is Hermitian if and only if AB commutes.

$(AB)^* = B^* A^* = BA$

If it commutes then

$(AB)^* = AB$

Real eigenvalues

Hermitian matrices have real eigenvalues.

$Hv = \lambda v$

$v^*Hv = \lambda v^*v$

$v^*Hv = \lambda$

Skew-Hermitian matrices

These are also known as anti-Hermitian matrices.

$M^* = -M$

If eigenvalues are different, eigenvectors are orthogonal

17.4.3 Pauli matrices

Pauli matrices are 2×2 matrices which are unitary and hermitian.

That is, $P^* = P^{-1}$.

And $P^* = P$.

The Pauli matrices

The matrices are:

$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

The identity matrix is often considered alongside these as:

$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Pauli matrices are their own inverse

$\sigma_i^2 = \sigma_i \sigma_i$

$\sigma_i^2 = \sigma_i \sigma_i^*$

$\sigma_i^2 = \sigma_i \sigma_i^{-1}$

$\sigma_i^2 = I$

Determinants and trace of Pauli matrices

$\det \sigma_i = -1$

$Tr(\sigma_i) = 0$

As the sum of eigenvalues is the trace, and the product is the determinant, the eigenvalues are 1 and -1.

17.4.4 Positive-definite matrices

The matrix M is positive definite if for all non-zero vectors the scalar is positive.

$v^T M v$

We know that the outcome is a scalar, so:

$v^T M v = (v^T M v)^T$

$v^T M v = v^T M^T v$

$v^T (M - M^T) v = 0$

17.4.5 Inner products

An inner product is a sesquilinear form with a positive-definite Hermitian matrix.

$\langle u, v\rangle = u^* H v$

If we are using the real field this is the same as:

$\langle u, v\rangle = u^T H v$

Where H is now a symmetric real matrix.

Same

$\langle v, v\rangle = v^* H v$

Always positive and real.

Properties

$\langle u, v\rangle\langle v, u\rangle = |\langle u, v\rangle|^2$

17.4.6 Cauchy-Schwarz inequality

This states that:

$|\langle u, v\rangle|^2 \leq \langle u, u\rangle \dot{\langle} v, v\rangle$

Consider the vectors u and v. We construct a third vector $u - \lambda v$. We know the length of any vector is non-negative. $0 \leq \langle u - \lambda v, u - \lambda v\rangle$

$0 \leq \langle u, u\rangle + \langle u, -\lambda v\rangle + \langle -\lambda v, u\rangle + \langle -\lambda v, -\lambda v\rangle$

$0 \leq \langle u, u\rangle - \bar{\lambda}\langle u, v\rangle - \lambda\langle v, u\rangle + \lambda\bar{\lambda}\langle v, v\rangle$

We now look for a value of λ to simplify this equation.

$\lambda = \frac{\langle u, v\rangle}{\langle v, v\rangle}$

$0 \leq \langle u, u\rangle - \frac{\langle v, u\rangle\langle u, v\rangle}{\langle v, v\rangle} - \frac{\langle u, v\rangle\langle v, u\rangle}{\langle v, v\rangle} + \frac{\langle u, v\rangle}{\langle v, v\rangle}\frac{\langle v, u\rangle}{\langle v, v\rangle}\langle v, v\rangle$

$0 \leq \langle u, u\rangle - \frac{|\langle u, v\rangle|^2}{\langle v, v\rangle}$

$|\langle u, v\rangle|^2 \geq \langle u, u\rangle\langle v, v\rangle$

17.4.7 The orthogonal projection

in inner product space, orthogonal projection

$p_u v = \frac{< u, v >}{v, v} v$

we then know that $o = v - p_u v$ is orthogonal to u.

17.4.8 Orthogonal set / Orthogonal basis

if a set of vectors are all orthogonal, they form an orthogonal set if the set spans the vector space, it is an orthogonal basis.

17.4.9 The Gram-Schmidt process

can we form an orthogonal basis from a non-orthogonal basis? yes, using gram schmidt

we have x_1, x_2 x_3 etc we want to make v_1, v_2 etc orthognal

$v_1 = x_1$ $v_2 = x_2 - p_{x_2} v_1$ $v_3 = x_3 - p_{x_3} v_1 - p_{x_3} v_2 v$

17.5 Special groups

17.5.1 Special orthogonal groups $SO(n, F)$

The special orthogonal group, $SO(n, F)$, is the subgroup of the orthogonal group where $|M| = 1$.

As a result it includes only the rotation operators, not the flip operators.

$SO(3)$ is rotations in 3d space.

$SO(2)$ is rotations in 2d space.

Determinant of the orthogonal group

The orthogonal group has determinants of -1 or 1.

$O^T = O^{-1}$

$\det(O^T) = \det(O^{-1})$

$\det O = \frac{1}{\det O}$

$\det O = \pm 1$

17.5.2 Special unitary groups $SU(n, F)$

The special unitary group, $SU(n, F)$, is the subgroup of $U(n, F)$ where the determinants are 1.

That is, $|M| = 1$

The determinant of unitary matrices

The determinant of the unitary matrices is:

$\det U^* = \det U^{-1}$

$(\det U)^* = \frac{1}{\det U}$

$(\det U)^* \det U = 1$

$|| \det U|| = 1$

17.5.3 Special lnear groups $SL(n, F)$

The special linear group, $SL(n, F)$, is the subgroup of $GL(n, F)$ where the determinants are 1.

That is, $|M| = 1$

These are endomorphisms, not forms.

17.6 Sort

17.6.1 Normal matrices

$M^*M = MM^*$

All symmetrix matrices are normal

All hermetitian matrices (inc subset symmetric) are normal

Normal matrix never defective

Chapter 18

Linear maps

18.1 Homomorphisms of vector spaces

18.1.1 Linear maps

Homomorphisms between vector spaces

Homomorphisms map between algebras, preserving the underlying structure.

A homomorphism vetween vector space V and vector space W can be described as:

$\hom(V, W)$

Homomorphism between vector spaces must preserve the group-like structure of the vector space.

$f(u+v) = f(u) + f(v)$

The homomorphism must also preserve scalar multiplication.

$f(\alpha v) = \alpha f(v)$

A linear map (or function) is a map from one input to an output which preserves addition and scalar multiplication.

That is if function f is linear then:

$f(aM + bN) = af(M) + bf(N)$

Alternative names for homomorphisms

Vector spaces homomorphisms are also called linear maps or linear functions.

18.1.2 Homomorphisms form a vector space

If we can can show that scalars can act on morphisms, then we can shwn that morphisms on a vector space are themselves a vector space.

Scalars can act on morphisms, and so morphisms of vector spaces are themselves vector spaces.

Dimensions of homomorphisms

We can identify the dimensionality of this new vector space from the dimensions of the original vector spaces.

$\dim(\hom(V, W)) = \dim V \dim W$

18.1.3 The pseudo-inverse

The definition of the inverse is that:

$MM^{-1} = I$

$M^{-1}M = I$

We also have:

$MM^{-1}M = M$

$M^{-1}MM^{-1} = M^{-1}$

The inverse of a homomorphism

Generally we don't have inverses of homomorphisms as the number of dimensions are different.

We can, however, find a matrix M^+ which satisfies:

$MM^+M = M$

$M^+MM^+ = M^+$

This is the pseudo-inverse.

18.1.4 Linear and affine functions

Linear maps

Linear maps can be written as:

$v = Mu$

These go through the origin. That is, if $u = 0$ then $v = 0$.

Affine function

Affine functions are more general than linear maps. They can be written as:

$v = Mu + c$

Where c is a vector in the same space as v.

Affine functions where $c \neq 0$ are not linear maps. They are not homomorphisms which preserve the structure of the vector space.

If we multiply u by a scalar s, then v will not increase by the same proportion.

18.1.5 Singular value decomposition

The singular value decomposition of $m \times n$ matrix M is:

$M = U\Sigma V^*$

Where:

- U is a unitary matrix $(m \times m)$
- Σ is a diagonal matrix with non-negative real numbers $(m \times n)$
- V is a unitary matrix $(n \times n)$

Σ is unique. U and V are not.

Properties

$M^*M = U\Sigma^2U^*$

$(M^*M)^{-1} = V\Sigma^{-2}V^*$

Calculating the SVD

The SVD is generally calculated iteratively.

18.1.6 Identity matrix and the Kronecker delta

The Kronecker delta

The Kronecker delta is defined as:

p$\delta_{ij} = 0$ where $i \neq j$

$\delta_{ij} = 1$ where $i = j$

We can use this to define matrices. For example for the identity matrix:

$I_{ij} = \delta_{ij}$

Identity matrix

A square matrix where every element is 0 except where $i = j$. There is one for each square matrix.

$$I = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix}$$

Chapter 19

Cross products and the Hodge star operator

19.1 Cross products

19.1.1 The cross product

$v \times u$

Cross product is a bilinear map

This is a bilinear map from two vectors in $\mathbb{R}^3$ to another vector in the same space.

$V \times V \to V$

Calculating the cross product

This is calcualted by:

$u \times v = ||u||||v|| \sin(\theta) n$

The resulting vector is perpendicular to both input vectors.

19.2 The Hodge star operator

19.2.1 Introduction

Chapter 20

Exterior algebra

20.1 Exterior algebra

20.1.1 The exterior (wedge) product

The exterior product of two vectors is:

$u \wedge v$

20.1.2 The exterior product is anticommutative

This is anticommutative (alternating).

$u \wedge v = -v \wedge u$

This implies that:

$u \wedge u = 0$

20.1.3 The exterior product is distributive

$(a + b) \wedge (c + d) = (a \wedge c) + (a \wedge d) + (b \wedge c) + (b \wedge d)$

20.1.4 Expanding the exterior product of two vectors

We can look at the exterior product in component-basis terms.

Consider 2-dimenional vector space with the following vectors:

$u = ae_1 + be_2$

$v = ce_1 + de_2$

The exterior product is:

$u \wedge v = (ae_1 + be_2) \wedge (ce_1 + de_2)$

$u \wedge v = (ae_1 \wedge ce_1) + (ae_1 \wedge de_2) + (be_2 \wedge ce_1) + (be_2 \wedge de_2)$

$u \wedge v = ac(e_1 \wedge e_1) + ad(e_1 \wedge e_2) + bc(e_2 \wedge e_1) + bd(e_2 \wedge e_2)$

$u \wedge v = ad(e_1 \wedge e_2) - bc(e_1 \wedge e_2)$

$u \wedge v = (ad - bc)(e_1 \wedge e_2)$

20.1.5 Exterior (Grassman) algebra

The exterior algebra is the algebra generated by the wedge product.

The term $u \wedge v$ can be interpreted as the area covered by the parallelogram generated by u and v.

As $a\mathbf{u} \wedge b\mathbf{v} = ab\mathbf{u} \wedge \mathbf{v}$, we can see that scaling the length of one of the vectors by a scalar, we also increase the exterior product by the same scalar.

20.1.6 Orientation

We can describe the exterior product of two vectors as $\mathbf{u} \wedge \mathbf{v}$ or $\mathbf{v} \wedge \mathbf{u}$.

20.1.7 Bivectors

20.1.8 Trivectors

Chapter 21

Tensors

21.1 Element-wise notation

21.1.1 Einstein summation convention

A vector can be written as a sum of its components.

$v = \sum_i e_i v^i$

The Einstein summation convention is to remove the $\sum_i$ symbols where they are implicit.

For example we would instead write the vector as:

$v = e_i v^i$

Adding vectors

$v + w = (\sum_i e_i v^i) + (\sum_i f_i w^i)$

$v + w = \sum_i (e_i v^i + f_i w^i)$

$v + w = e_i v^i + f_i w^i$

If the bases are the same then:

$v + w = e_i (v^i + w^i)$

Scalar multiplication

$cv = c \sum_i e_i v^i$

$cv = \sum_i c e_i v^i$

$cv = c e_i v^i$

Matrix multiplication

$AB_{ik} = \sum_j A_{ij} B_{jk}$

$AB_{ik} = A_{ij} B_{jk}$

Inner products

$\langle v, w \rangle = \langle \sum_i e_i v^i, \sum_j f_j w^j \rangle$

$\langle v, w \rangle = \sum_i v^i \langle e_i, \sum_j f_i w^j \rangle$

$\langle v, w \rangle = \sum_i \sum_j v^i \overline{w^j} \langle e_i, f_j \rangle$

If the two bases are the same then:

$\langle v, w \rangle = \sum_i \sum_j v^i \overline{w^j} \langle e_i, e_j \rangle$

We can define the metric as:

$g_{ij} := \langle e_i, e_j \rangle$

$\langle v, w \rangle = v^i \overline{w^j} g_{ij}$

21.1.2 Covariant and contravariant bases

In element form we write a vector as:

$v = e_i v^i$

The indices are raised and lowered to reflect whether the value is covariant or contravariant.

v^i is contravariant. If the basis moves one way, it moves the other.

e_i is covariant. If the basis moves, it moves with it.

21.2 Tensor product

21.2.1 Tensor product

We have spaces V and W over field F. If we have a linear operation which takes a vector from each space and returns a scalar from the underlying field, it is an element of the tensor product of the two spaces.

For example if we have two vectors:

$v = e_i v^i$

$w = e_j w^j$

A tensor product would take these and return a scalar.

There are three types of tensor products:

- Both are from the vector space
- $T_{ij}v^iw^j$
- $T_{ij} \in V \otimes W$
- Both are from the dual space
- $T^{ij}v_iw_j$
- $T_{ij} \in V^* \otimes W^*$
- One is from each space
- $T_i^jv^iw_j$
- $T_{ij} \in V \otimes W^*$

As a vector space, we can add together tensor products, and do scalar multiplication.

Basis of a tensor product

Not all elements spanned by a basis of a tensor product are themselves tensor products.

Eigenvalues and Eigenvectors of a tensor product

Homomorphisms

We can define homomorphisms in terms of tensor products.

$Hom(V) = V \otimes V^*$

T_j^i

We use the dual space for the second argument. This is because it ensures that changes to the bases do not affect the maps.

$w^j = T_i^jv^i$

21.2.2 Raising and lowering indices

We showed that the inner product between two vectors with the same basis can be written as:

$\langle v, w\rangle = \langle \sum_i e_iv^i, \sum_j f_jw^j\rangle$

$\langle v, w\rangle = v^i\overline{w^j}\langle e_i, e_j\rangle$

Defining the metric as:

$g_{ij} := \langle e_i, e_j\rangle$

$\langle v, w\rangle = v^i\overline{w^j}g_{ij}$

Metric inverse

We can use this to define the inverse of the metric.

$g^{ij} := (g_{ij})^{-1}$

We can use this to raise and lower vectors.

$v_i := v^j g_{ij}$

Raising and lowering indices of tensors

If we have tensor:

T_{ij}

We can define:

$T_i^k = T_{ij} g^{jk}$

$T^{il} = T_{ij} g^{jk} g^{kl}$

Tensor contraction

If we have:

$T_{ij} x^j$

We can contract it to:

$T_{ij} x^j = v_i$

Similarly we can have:

$T^{ij} x_j = v^i$

21.2.3 Kronecker delta

Consider matrix multiplication AI.

We have:

$AI_{ik} = A_{ij} I_{jk}$

We write this instead as:

$AI_{ik} = A_{ij} \delta_{jk}$

Where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$.

21.2.4 Tensors form a vector space

Recap

Tensors form a vector space

Dimension of a tensor

Basis of a tensor

21.3 Tensors

21.3.1 Tensor valence

21.3.2 Tensor inverses

For second order tensors we have:

- T^i_j
- T_{ij}
- T^{ij}

For each of these we can define an inverse:

- $T^j_i U^k_j = \delta^k_i$
- $T_{ij} U^{jk} = \delta^k_i$
- $T^{ij} U_{jk} = \delta^k_i$

Notation for inverses

If we have $T_{ij} U^{jk} = \delta^k_i$, we can instead write:

$T_{ij} T^{jk} = \delta^k_i$

21.3.3 Tensor contraction

We have a vector $v \in V$ and $w \in V^*$.

$\mathbf{v} = \sum_i v^i \mathbf{e}_i$

$\mathbf{w} = \sum_i w_i \mathbf{f}^i$

$\mathbf{wv} = [\sum_i v^i \mathbf{e}_i][\sum_i w_i \mathbf{f}^i]$

$\mathbf{wv} = \sum_i \sum_j [v^i \mathbf{e}_i][w_j \mathbf{f}^j]$

$\mathbf{wv} = \sum_i \sum_j v^i w_j \mathbf{e}_i \mathbf{f}^j$

We use the dual basis so:

$\mathbf{wv} = \sum_i \sum_j v^i w_j \mathbf{e}_i \mathbf{e}^j$

$\mathbf{wv} = \sum_i \sum_j v^i w_j \delta_i^j$

We can see that this value is unchanged when there is a change in basis.

What if these were both from V?

$\mathbf{v} = \sum_i v^i \mathbf{e}_i$

$\mathbf{w} = \sum_i w^i \mathbf{e}_i$

$\mathbf{wv} = [\sum_i v^i \mathbf{e}_i][\sum_i w^i \mathbf{e}_i]$

$\mathbf{wv} = \sum_i \sum_j v^i w^j \mathbf{e}_i \mathbf{e}_i$

This term is dependent on the basis, and so we do not contract.

So if we have $v_i w^i$, we can contract, because the result (calculated from the components) does not depend on the basis.

But if we have $v_i w_i$, the result (calcualted from the components) will change depending on the choice of basis.

We define a new object

$c = \sum_i w^i v_i$

This new term, c, does not depend on i, and so we have contracted the index.

21.3.4 Symmetric and antisymmetric tensors

Consider a tensor, e g T_{abc}.

In general, this is not symmetric, that is:

$T_{abc} \neq T_{bac}$

Symmetric part of a tensor

We can write the symmetric part of this with regard to a and b.

$T_{(ab)c} = \frac{1}{2}(T_{abc} + T_{bac})$

Clearly, $T_{(ab)c} = T_{(ba)c}$

Antisymmetric part of a tensor

We can also have an an antisymmetric part with regard to a and b.

$T_{[ab]c} = \frac{1}{2}(T_{abc} - T_{bac})$

Clearly, $T_{[ab]c} = -T_{[ba]c}$

Tensors as sums of their symmetric and antisymmetric parts

$T_{(ab)c} + T_{[ab]c} = \frac{1}{2}(T_{abc} + T_{bac}) + \frac{1}{2}(T_{abc} - T_{bac})$

$T_{(ab)c} + T_{[ab]c} = T_{abc}$

21.4 Higher-order tensors

21.4.1 Higher-order tensors

We can create higher order tensors products. For example

$V \otimes V \otimes V \otimes V^* \otimes V^*$

We write elements of these as:

$T^{i_1,...,i_p}_{j_1,...,j_q}$

We can map from matrix to matrix etc higher dimensional

Matrix has A: a_{ij}.

Tensor can have T: t_{ijk} for example

0 rank tensor: scalar

1 rank tensor: vector

2 rank tensor: matrix

page on covariance and contravariance and type (p, q)

21.5 Sort

21.5.1 Outer product

The outer product is a bilinear map

This is a bilinear map from two vectors from the same vector space to another vector space.

$V \times V \to V$

Calculating the outer product

$u \otimes v = w$

$w_{ij} = u_i v_j$

The dimensions of the tensor outer product

$\dim(V \otimes W) = \dim V \times \dim W$

Outer product on the complex numbers

Relation between the dot product and outer product

The dot product in the trace of the outer product.

21.5.2 Kronecker product

The Kronecker product takes the concept of the outer product and applies to to matrices.

We can essentially repace every element in the matrix on the left with the element multiplied by the entire matrix on the right.

Like outer products, Kronecker products are written as:

$u \otimes v = w$

21.5.3 Dot product

Dot product is a bilinear form

This is a bilinear form, a mapping from two vectors in the same vector space to the underlying field.

$V \times V \to F$

Calculating the dot product

This is calculated by multiplying each matching element, and summing the results.

$u \cdot v = \sum_{i=1}^n u_i v_i$

Dot product on the complex numbers

Properties don't hold. Can get zero vectors from non-zero inputs. Get complex numbers from dot product on itself.

Inner products better deal with complex number fields. However they are not bilinear maps.

21.5.4 Homomorphism as a tensor product

21.5.5 Tensors

A tensor is an element of a tensor product.