Geometry

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Preface

This is a live document, and is full of gaps, mistakes, typos etc.

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Points, lines and affine transformations

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Chapter 2

Euclidian transformations, lengths and angles

2.1 Linear metrics

2.1.1 Metrics

We defined a norm as:

$||v|| = v^T M v$

A metric is the distance between two vectors.

$d(u, v) = ||u - v|| = (u - v)^T M (u - v)$

Metric space

A set with a metric is a metric space.

2.1.2 Inducing a topology

Metric spaces can be used to induce a topology.

2.1.3 Translation symmetry

The distance between two vectors is:

$(v - w)^T M (v - w)$

So what operations can we do now?

As before, we can do the transformations which preserve $u^T Mv$, such as the orthogonal group.

But we can also do other translations

$(v-w)TM(v-w)$

$v^T Mv + w^T Mw - v^T Mw - w^T Mv$

so symmetry is now $O(3,1)$ and affine translations

Translation matrix

$[[1,x][0,1]]$ moves vector by x.

2.2 Specific groups

2.2.1 The affine group

2.2.2 The Euclidian group

2.2.3 The Galilean group

2.2.4 The Poincar group

2.3 Non-linear norms

2.3.1 L_p norms (p-norms)

L^P norm

This generalises the Euclidian norm.

$||x||_p = (\sum_{i=1}^n |x|_i^p)^{1/p}$

This can defined for different values of p. Note that the absolute value of each element in the vector is used.

Note also that:

$||x||_2$

Is the Euclidian norm.

Taxicab norm

This is the L^1 norm. That is:

$||x||_1 = \sum_{i=1}^{n} |x|_i$

Angles

Cauchy-Schwarz

2.4 To linear forms

2.4.1 Norms

We can use norms to denote the "length" of a single vector.

$||v|| = \sqrt{\langle v, v \rangle}$

$||v|| = \sqrt{v^* M v}$

Euclidian norm

If $M = I$ we have the Euclidian norm.

$||v|| = \sqrt{v^* v}$

If we are using the real field this is:

$||v|| = \sqrt{\sum_{i=1}^{n} v_i^2}$

Pythagoras' theorem

If $n = 2$ we have in the real field we have:

$||v|| = \sqrt{v_1^2 + v_2^2}$

We call the two inputs x and y, and the length z.

$z = \sqrt{x^2 + y^2}$

$z^2 = x^2 + y^2$

2.4.2 Angles

Recap: Cauchy-Schwarz inequality

This states that:

$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\dot{\langle} v, v\rangle$

Or:

$\langle v, u\rangle\langle u, v\rangle \leq \langle u, u\rangle\dot{\langle} v, v\rangle$

Introduction

$\langle v, u\rangle\langle u, v\rangle \leq \langle u, u\rangle\dot{\langle} v, v\rangle$

$\frac{\langle v, u\rangle\langle u, v\rangle}{||u||.||v||} \leq ||u||.||v||$

$\frac{||u||.||v||}{\langle v, u\rangle} \geq \frac{\langle u, v\rangle}{||u||.||v||}$

$\cos(\theta) = \frac{\langle u, v\rangle}{||u||.||v||}$

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4.1 Elementary geometry in 2 dimensions

4.1.1 Triangles

Area of a triangle

Circumference of a triangle

Sum of angles of a triangle

Angles in a triangle add to π.

4.1.2 Quadrilaterals

4.1.3 Oblongs

Area of an oblong

Circumference of an oblong

4.1.4 Squares

Area of a square

$A = r^2$

Circumference of a square

$C = 2r$

Angles in a square

Angles in a square sum to 2π.

4.1.5 Pentagon

Chapter 5

3D polygons

5.1 Elementary geometry in 3 dimensions

5.1.1 Pyramid

5.1.2 Cubes

Volume of a cube:

$V = r^3$

Surface area of a cube:

$A = 6r^2$

Chapter 6

Algebraic geometry and spheres

6.1 Circles

6.1.1 Defining circles

$x^2 + y^2 = r^2$

6.1.2 Area of a circle

$A = \pi r^2$

6.1.3 Circumference of a circle

$C = 2\pi r$

6.2 Spheres

6.2.1 Defining spheres

$x^2 + y^2 + z^2 = r^2$

6.2.2 Volume of a sphere

$V =$

6.2.3 Surface area of a sphere

$A =$

Part II

Abstract algebra

Chapter 7

Group theory

7.1 Introduction

7.1.1 Abstract algebra

Abstract algebra allows us to discuss properties of types of mathematical structures.

Rather than construct a specific object, and explore its properties, we can explore the properties of an abstract structure with certain definitions. We can then apply findings from this to an any structure which meets the definition.

Examples of abstract algebra

We explore:

- Groups
- Rings
- Fields
- Vector spaces
- Inner product spaces

7.1.2 Defining groups

Magma

A magma, or groupoid, is a set with a single binary operation.

These can be defined as an ordered pair $(s, \odot)$ where s is the set, and $\odot$ is the binary operation.

If a and b are in s, then $a \odot b$ is also in s.

The following are magmas:

- Natural numbers and addition
- $n \times n$ matrices with determinants other than 0
- Natural numbers above 0 and addition
- Integers and addition
- Rational numbers and division
- $\{-1, 1\}$ and multiplication

The following are not magmas:

- Natural numbers up to 10 and addition

Semigroup

A semigroup is a magma whose binary operation is associative.

The following are semigroups:

- Natural numbers and addition
- $n \times n$ matrices with determinants other than 0
- Natural numbers above 0 and addition
- Integers and addition

The following are not semigroups:

- $\{-1, 1\}$ and multiplication
- Rational numbers and division
- Natural numbers up to 10 and addition

Monoid

A monoid is a semigroup with an identity element

The following are monoids:

- Natural numbers and addition
- $n \times n$ matrices with determinants other than 0
- Integers and addition

- $\{-1, 1\}$ and multiplication

The following are not monoids:

- Natural numbers above 0 and addition
- Rational numbers and division
- Natural numbers up to 10 and addition

Group

A group is a monoid where there is an inverse operation for the binary operation.

The following are groups:

- Integers and addition
- $n \times n$ matrices with determinants other than 0
- $\{-1, 1\}$ and multiplication

The following are not groups:

- Natural numbers above 0 and addition
- Rational numbers and division
- Natural numbers and addition
- Natural numbers up to 10 and addition

7.1.3 Subgroups

A subgroup of a group is a subset of a group, which also forms a group with the same element.

For example all even numbers are a subgroup of the addition group of integers.

7.1.4 Abelian groups

A commutative group, that is where $a \odot b = b \odot a$.

The following are abelian groups:

- Integers and addition
- $\{-1, 1\}$ and multiplication

The following are not abelian groups:

- Natural numbers above 0 and addition

- Rational numbers and division
- Natural numbers and addition
- Natural numbers up to 10 and addition
- $n \times n$ matrices with determinants other than 0

7.1.5 Group order

For finite groups, each element e has:

$e^n = I$

For some $n \in \mathbb{N}$

Where I is the identity element.

The order of the group is the smallest value of n such that that holds for all elements.

For example in the multiplicative group $G = \{-1, 1\}$ the order is 2.

Or:

$|G| = 2$

Additionally

$|-1| = 2$

$|1| = 1$

7.2 Creating groups

7.2.1 Permutations and the symmetric group

A permutation is defined as a bijection from a set to itself.

For a set of size n, the number of permutations is $n!$. This is because there are n possibilities for the first item, $n-1$ for the second and so on.

The symmetric group

The set of all permutations forms a group, the symmetric group. This forms a group because:

- There is an identity element
- Each combination of permutations is also in the group.

- Each permutation has an inverse in the group.

Permutation groups

A subgroup of the symmetric group is called a permutation group.

7.2.2 Morphism

Morphisms are functions which preserve the relationships between members of a set, and specified functions.

That is, if:

$a \odot b = c$

Then $f(x)$ is morphism if:

$f(a) \odot f(b) = f(a \odot b)$

Here we discuss morphisms in the context of groups, but we can define morphisms for sets with more than one function, for example with addition and multiplication.

Morphisms are also known as homomorphisms.

The following are morphisms of the additive group of integers.

Where we refer to c, $c \neq 0 \in \mathbb{I}$.

- $f(x) = 0$
- $f(x) = x$
- $f(x) = cx$
- Converting natural numbers to integers

The following are not morphisms

- $f(x) = x + 1$

Isomorphism

An isomorphism is a morphism which has an inverse.

This means the function is bijective.

The following are isomorphisms:

- $f(x) = x$
- $f(x) = cx$

- Converting natural numbers to integers

The following are not isomorphisms

- $f(x) = 0$
- $f(x) = x + 1$

Endomorphism

An endomorphism is one where the domain and codomain are the same.

The following are endomorphisms:

- $f(x) = 0$
- $f(x) = x$
- $f(x) = cx$

The following are not endomorphisms

- Converting natural numbers to integers
- $f(x) = x + 1$

Automorphism

An endomorphism which is also an isomorphism

The following are automorphisms:

- $f(x) = x$
- $f(x) = cx$

The following are not automorphisms

- $f(x) = 0$
- $f(x) = x + 1$
- Converting natural numbers to integers

Monomorphism

A morphism which is injective. That is:

$f(a) = f(b) \rightarrow a = b$

The following are monomorphisms:

- $f(x) = x$

- $f(x) = cx$
- Converting natural numbers to integers

The following are not monomorphisms:

- $f(x) = 0$
- $f(x) = x + 1$

7.2.3 Generating sets

We can define a group through a generating set and an operation.

And define the group as $G =< S >$

7.2.4 Finite groups

Consider the set of natural numbers and addition modulo 4. This forms a group containing:

$\{0, 1, 2, 3\}$

This can be written as Z_4 or more generally as Z_n, or Z/nZ.

7.3 Group operations

7.3.1 The group commutator

The group commutator is:

$[a, b] = a^{-1}b^{-1}ab$

If the group is abelian then $[a, b] = 0$. The group commutator is a measure of how non-abelian the group is.

This has the following properties:

- Alternativity: $[A, A] = I$

7.3.2 The direct product of groups

If we have two groups G and H we can form new group $G \times H$.

For every $g \in G$ and $h \in H$ there is $(g, h) \in G \times H$.

The binary operation we have is:

$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1, h_2)$

7.4 Specifc groups

7.4.1 The trivial group

The trivial group is the group with just the identity member I.

7.4.2 The infinite cyclic group (Z)

The additive group of integers

Generating cyclic groups

We can generate a group with a single element, it is a cyclic group.

For example, we can define a group $G =< 1 >$ which gives us the additive group of integers.

Infinite cyclic groups are isomorphic to the additive group of integers

More generally, any infinite cyclic group is isomorphic to the additive group of integers.

Consider the multiplicative group of $< i >$.

This contains $\{1, -1, i, -i\}$.

This is also automorphic to the natural number and modulo addition group above.

We can define finite cyclic groups of size n using the generating element $z^{\frac{1}{n}}$. This is isomorphic to the general cyclic group C_n, and to Z/nZ.

Abelian cyclic groups

Cyclic groups are abelian.

7.4.3 The finite cyclic groups (C_n or Z_n)

7.4.4 The circle group T

The circle group, T, includes all complex numbers of magnitude 1.

7.5 Normal subgroups

7.5.1 Cosets and normal subgroups

A coset is defined between a group and a subgroup of the group.

For a group G, and its subgroup H:

- The left coset is $\{gH\}$
- The right coset is $\{Hg\}$

For $\forall g \in G$.

For abelian groups, the left and right cosets are the same.

The left and right cosets can also be the same, even if the group G is not abelian.

Normal subgroups

If the left and right cosets are the same then H is a normal subgroup.

Cosets divide a group.

Consider two left cosets, aH and bH, with a common element.

This means that $ah_i = bh_j$.

We can use this to get:

$a = bh_j h_i^{-1}$

$b = ah_i h_j^{-1}$

We know that:

$ah \in aH$

$bh \in bH$

So:

$bh_j h_i^{-1} h \in aH$

$ah_ih_j^{-1}h \in bH$

And so:

$bH \subset aH$

$aH \subset bH$

Therefore:

$aH = bH$

Example 1

Consider the group $\{-1, 1\}, \times$

For the subgroup $\{1\}, \times$, the left coset is $\{gH\} = \{1, -1\}$.

The right coset is the same.

Example 2

Consider the group of integers and addition: $(Z, +)$

For subgroup $(mZ, +)$, the left and right cosets are the same because the group is abelian.

The coset of the subgroup is the subgroup multiplied by each element in G.

This is mZ, $mZ + 1$, $mZ + 2$ and so on.

Once we reach $mZ + m$ this has looped, and is already a coset, so we only need the sets upto $mZ + m - 1$.

7.5.2 Quotient groups

We have a group G and a normal subgroup N.

We define a quotient group from this as G/N. This is the set of cosets from N.

7.5.3 Group extension

This defines a group G from a normal subgroup N and a quotient group Q.

7.6 Theorems

7.6.1 Cayley's theorem

Cayley's theorem states that every group G is isomorphic to a subgroup of the symmetric group acting on G.

Multiplication by a member of G is a bijective function, as for each g there is also a g^{-1}.

This means that multiplication of each member of G is a permutation, and so is a subset of the symmetric group on G.

7.6.2 Lagrange's theorem

Lagrange's theorem states that for any finite group G, the order of every subgroup is a divisor of the order of G.

Consider subset H. We know that all cosets are disjoint, and that the union of all cosets is G.

As cosets are the same size, we know that:

$|G| = m|H|$, where m is the number of cosets.

This means that if a group has order 10, a subgroup must have order 1, 2 5 or 10.

7.7 Group action

7.7.1 Group action

We have a group G and a set S.

We have a function $g.s$ which maps onto S such that:

- $I.s = s$
- $(gh).s = g(h.s)$

Chapter 8

Rings

8.1 Introduction

8.1.1 Rings

Consider an abelian group $(S, +)$.

A ring takes this and adds a multiplicative function which satisfies the distributive property.

Groups have an identity element for their function. Rings must have identity elements for both their functions.

The multiplicative function does not have to be a bijection. For example the set of integers, addition and multiplication form a ring.

8.1.2 Rngs

A rng is a ring without the multiplicative identity (hense no 'i').

8.2 Commutation

8.2.1 Commutative rings

The multiplication operation commutes.

8.2.2 Commutator

$[a, b] = ab - ba$

8.2.3 The Jacobi identity

8.3 Examples of rings

8.3.1 Zero (trivial) ring

The trivial ring is a ring with just one element.

0 with addition and multiplication work.

8.3.2 Integer rings

The integers with addition and multiplication form a ring.

8.3.3 Integer mod n rings

The integers mod n with addition and multiplication form a group.

Examples

The integers $\{1, 2, 3\}$ form a ring.

8.4 Properties of rings

8.4.1 Characteristic of a ring

The characteristic of a ring is the number of times the multiplicative identity must be added to get the additive identity.

If this never happens, the characteristic is 0.

Example

The integer mod 2 ring, the characteristic is 2.

8.5 Division

8.5.1 Division rings

A divison ring is a ring where every non-zero element has a multiplicative inverse.

Example

The rational numbers are a divison ring.

Relationship between divison rings and fields

Fields (not yet introduced) are different from division rings only in that multiplication for a field must be commutative.

8.5.2 Units

A unit is an element of a ring which has a multiplicative inverse.

Examples

The ring of integers with addition and multiplication, only -1 and 1 are units, as both have multiplicative inverses in the ring.

8.6 Subrings

8.6.1 Subrings

A subring is a subset of the ring, where the addition and multiplication operations on the subring result in elements also in the subring.

Example

The even numbers are a subring of the integers.

8.6.2 Ideals

An ideal is a subring where the multiplication of any element of the ideal with any element of the ring is also in the ideal.

Examples

Even numbers are an ideal of the integers.

Odd numbers are not an ideal. For example 1 is in the ideal, but multiplied by 2 gives 2, which is not in the ideal.

Chapter 9

Fields

9.1 Fields

9.1.1 Fields

A field is a ring where the multiplication function has an inverse.

The integers, addition and multiplication form a ring, but not a group.

The rational numbers (except 0), addition and multiplication form a field (and a ring).

The real numbers and complex numbers also form fields.

9.1.2 Finite (Galois) fields

Finite number of elements.

Integers mod p field

Characteristic of a field

9.2 Algebra on a field

9.2.1 Bilinear maps

A bilinear map (or function) is a map from two inputs to an output which preserves addition and scalar multiplication. This is in contrast to a linear map, which only has one input.

In addition, the function is linear in both arguments.

That is if function f is bilinear then:

$X = aM + bN$

$Y = cO + dP$

$f(X, Y) = f(aM + bN, cO + dP)$

$f(X, Y) = f(aM, cO + dP) + f(bN, cO + dP)$

$f(X, Y) = f(aM, cO) + f(aM, dP) + f(bN, cO) + f(bN, dP)$

$f(X, Y) = acf(M, O) + adf(M, P) + bcf(N, O) + bdf(N, P)$

Note that:

$f(X, Y) = f(X + 0, Y)$

$f(X, Y) = f(X, Y) + f(0, Y)$

$(0, Y) = 0$

That is, if any input is 0 in an additative sense, the value of the map must be zero.

9.2.2 Algebra on a field

Part III

Abstract linear algebra

Chapter 10

Vector spaces

10.1 Vector spaces

10.1.1 Vector spaces

A vector space is a group with additional structure.

The operation for each element is shown as addition. So we can say:

$\forall u, v \in V[u + v \in V]$

To this we add scalars, from a field F. We write this as multiplication.

$\forall f \in F \forall v \in V[fv \in V]$

Subspace

A subspace is a subset of V which still acts as a vector space. In practice, this means fewer dimensions.

10.1.2 Span

Span function

We can take a subset S of V. We can then make linear combinations of these elements.

This is called the linear span - $span(S)$.

10.1.3 Linear dependence

A collection of vectors in a vector space are linearly dependent if there exist values for α (other than all being 0) such that:

$\sum_i \alpha_i v_i = 0.$

If no such values for α exist we say the vectors are linearly independent.

10.1.4 Basis vectors

Basis

We can write vectors as combinations of other vectors.

$v = \sum_i \alpha_i v_i$

A subset which spans the vector space, and which is also linearly independent, is a basis of the vector space.

For an arbitrary vector of size n, we cannot use less than n elementary vectors. We could use more, but these would be redundant.

If we use n elementary vectors, there is a unique solution of weights of elementary vectors.

If we use more than n elementary vectors, there will be linear dependence, and so there will not be a unique solution.

10.1.5 Dimension function

For a basis S, the the dimension of the vector space is $|S|$.

$\dim(V) = |S|$

$S \subset V$

Finite and infinite vector spaces

If $\dim(V)$ is finite, then we say the vector space is finite.

Otherwise, we say the vector space is infinite.

10.2 Points, lines and planes

10.2.1 Points, lines and planes

$(1, 0)$ is point, $(x, 2x + 1)$ is a line $(1, x, y)$ is a plane

10.2.2 Parallel lines and planes

Parallel lines

If we have two lines:

Parallel planes

Chapter 11

Linear endomorphisms

11.1 Endomorphisms of vector spaces

11.1.1 Endomorphisms

An endomorphism maps a vector space onto itself.

$end(V) = \hom(V, V)$

11.1.2 Endomorphisms form a vector space

An endomorphism maps a vector space onto itself.

$end(V) = \hom(V, V)$

Need to show that endomorphism is a vector space

Essentially

$v \in V$

$f \in F$

$av = f$

$bv = g$

$(a \oplus b)v = f + g$

$(a \oplus b)v = av + bv$

so there is some operation we can do on two members of endo linear in addition. That is, if we have two dual "things", we can define the addition of functions as the operation which results int he outputs being added.

what about linear in scalar? same approach.

Well we define

$c \odot a) = cav$

There is a unique endomorphism which results in two other endomorphisms being added together. define this as addition

11.1.3 Dimension of endomorphisms

$\dim(end(V)) = (\dim V)^2$

11.1.4 Basis of endomorphisms

11.1.5 Projections

A projection is a linear map which if applied again returns the original result.

A projection can drop a dimension for example.

11.1.6 Kernels and images

The kernel of a linear operator is the set of vectors such that:

$Mv = 0$

The kernel is also called the nullspace.

This can be shown as $\ker(M)$

The image of a linear operator is the set of vectors w such that:

$Mv = w$.

This can be shown as $\Im(M)$

We also know that:

$span(M) = \ker(M) + \Im(M)$

11.2 Representing endomorphisms with matrices

11.2.1 Matrix representation

Representing linear maps as matrices

We previously discsussed morphisms on vector spaces. We can write these as matrices.

Matrices represents transformations of vector spaces

Representing vectors as matrices

We can represent vectors as row or column matrices.

$v = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix}$

$v = \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_m \end{bmatrix}$

11.3 Automorphisms of vector spaces

11.3.1 Basis of an endomorphism

11.3.2 Changing the basis

For any two bases, there is a unique linear mapping from of the element vectors to the other.

11.4 The linear groups

11.4.1 General linear groups $GL(n, F)$

The general linear group, $GL(n, F)$, contains all $n \odot n$ invertible matrices M over field F.

The binary operation is multiplication.

11.4.2 Endomorphisms as group actions

We can view each member of the group g as a homomorphim on s.

Where s is a vector space V, the representation on each group member is an invertible square matrix.

If the set we use is the vector space V, then we can represent each group element with a square matrix acting on V.

Faithful means $a \neq b$ holds for repesentation too.

Representation theory. groups defined by $ab = c$. if we can match each eleemnt to amatrix where this holds we have represented the matrix.

11.4.3 Representing finite groups

Finite groups can all be represented with square matrices.

11.4.4 Representing compact groups

Chapter 12

Linear forms

12.1 Linear forms

12.1.1 Linear forms

A linear form is a linear map from a vector space to a scalar from the vector space's underlying field.

$\hom(V, F)$

Matrix operators

Linear forms can be represented as matrix operators.

$v^T M = f$

Where M has only one column.

Stuff

$f(M) = f(v)$

We introduce e_i, the element vector. This is 0 for all entries except for i where it is 1. Any vector can be shown as a sum of these vectors multiplied by a scalar.

$f(M) = f(\sum_{i=1}^m a_i e_i)$

$f(M) = \sum_{i=1}^m f(a_i e_i)$

$f(M) = \sum_{i=1}^m a_i f(e_i)$

$f(M) = \sum_{i=1}^m a_i f(e_i)$

Orthonormal basis

$f(M) = \sum_{i=1}^{m} a_i$

12.1.2 Dual space

The dual space V^* of vector space V is the set of all linear forms, $\hom(V, F)$.

The dual space is itself a vector space

$v \in V$

$f \in F$

$av = f$

$bv = g$

$(a \oplus b)v = f + g$

$(a \oplus b)v = av + bv$

So there is some operation we can do on two members of dual space

Linear in addition. That is, if we have two dual "things", we can define the addition of functions as the operation which results int he outputs being added.

what about linear in scalar? same approach.

Well we define

$(c \odot a) = cav$

The dual space has the same dimension as the underlying vector space

12.1.3 The dual space forms a vector space

The dual space forms a vector space. We can define addition and scalar multiplication on members of the dual space.

The dimension of the dual space is the same as the underlying space.

We have defined the dual space. A vector in dual space will have also have components and a basis.

$\mathbf{w} = \sum_i w_i f^j$

So how we describe the components will depend on the choice of basis.

We choose the dual basis, the basis for V^* as:

$\mathbf{e}_i\mathbf{f}^j = \delta_i^j$

If the basis changes, so does the dual basis.

We write the dual basis as e^j

12.2 Bilinear forms

12.2.1 Bilinear forms

A bilinear form takes two vectors and produces a scalar from the underyling field.

This is in contrast to a linear form, which only has one input.

In addition, the function is linear in both arguments.

$\phi(au+x, bv+y) = \phi(au, bv) + \phi(au, y) + \phi(x, bv) + \phi(x, y)$

$\phi(au+x, bv+y) = ab\phi(u, v) + a\phi(u, y) + b\phi(x, v) + \phi(x, y)$

Representing bilinear forms

They can be represented as:

$\phi(u, v) = v^T M u$

$f(M) = f([v_1, v_2])$

We introduce e_i, the element vector. This is 0 for all entries except for i where it is 1. Any vector can be shown as a sum of these vectors multiplied by a scalar.

$f(M) = f([\sum_{i=1}^m a_{1i}e_i, \sum_{i=1}^m a_{2i}e_i])$

$f(M) = \sum_{k=1}^m f([a_{1k}e_k, \sum_{i=1}^m a_{2i}e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m f([a_{1k}e_k, a_{2i}e_i])$

Because this in linear in scalars:

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}a_{2i}f([e_k, e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}a_{2i}e_k^T M e_i$

Orthonormal basis and $M = I$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}a_{2i}e_k^T e_i$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}a_{2i}\delta_i^k$

$f(M) = \sum_{i=1}^m a_{1i}a_{2i}$

12.2.2 The dot product

$v^T M u = f$

If the operator is I then we have the dot product.

$v^T u$

12.2.3 Orthogonal vectors

Given a metric M, two vectors v and u are orthogonal if:

$v^T M u = 0$

For example if we have the metric $M = I$, then two vectors are orthogonal if:

$v^T u = 0$

12.2.4 Metric-preserving transformations and isometry groups

If we have a bilinear form we can write the form as:

$u^T M v$

After a transformation P to the vectors it is:

$(Pu)^T M (Pv)$

$u^T P^T M P v$

So the value of the metric will be unaffected if:

$u^T P^T M P v = u^T M v$

$P^T M P = M$

Equivalent metrics

Different metrics can produce the same group. For example multiplying the metric by a constant.

$P^T M P = M$

12.2.5 Orthogonal groups $O(n, F)$

Recap: Metric-preserving transformations

The bilinear form is:

$u^T Mv$

The transformations which preserve this are:

$P^T MP = M$

The orthogonal group

If the metic is $M = I$ then the condition is:

$P^T P = I$

$P^T = P^{-1}$

These form the orthogonal group.

We use O instead of P:

$O^T = O^{-1}$

Rotations and reflections

The orthogonal group is the rotations and reflections.

Parameters of the orthogonal group

The orthogonal group depends on the dimension of the vector space, and the underlying field. So we can have:

- $O(n, R)$; and
- $O(n, C)$.

We generally refer only to the reals

$O(n)$ means $O(n, R)$.

The generally refer to the reals only.

12.2.6 Indefinite (pseduo) and split orthognal groups $O(n, m, F)$

Recap: Metric-preserving transformations

The bilinear form is:

$u^T Mv$

The transformations which preserve this are:

$P^TMP=M$

The metric

If the metric is:

$$M=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Then we have the indefinite orthogonal group $O(3,1)$

The split orthogonal group

Where $n=m$ we have the split orthogonal group.

$O(n,n,F)$

Signatures

12.2.7 The Lorentz group

The Lorentz group is the $O(1,3)$ group.

Symmetries of the Lorentz group

We can do the usual 3 rotations, however there are additional 3 symmetries, making the Lorzentz group 6-dimensional.

These are the Lorentz boosts.

A symmetry has:

$t'^2-x'2-y'^2-z'^2=t^2-x^2-y^2-z^2$

We consider the case where we just boost on x, so $y=y'$ and $z=z'$.

$t'^2-x'2=t^2-x^2$

Or with c:

$c^2t'^2-x'2=ct^2-x^2$

New

$s^2 = t^2 - x^2 - y^z - z^2$

$s'^2 = t'^2 - x'^2 - y'^2 - z'^2$

$ds^2 = s'^2 - s^2$

$ds^2 = (t'^2 - x'^2 - y'^2 - z'^2) - (t^2 - x^2 - y^z - z^2)$

$ds^2 = (t'^2 - t^2) - (x'^2 - x^2) - (y'^2 - y^2) - (z'^2 - z^2)$

$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$

boost: $s^2 = c^2t^2 - x^2 - y^z - z^2$

we want new t and x where distance is same $c^2t'^2 - x'^2 - y^z - z^2 = c^2t^2 - x^2 - y^z - z^2$
$c^2t'^2 - x'^2 = c^2t^2 - x^2$

We know that both transformations are linear [WHY??], therefore $x' = Ax + Bt$
$t' = Cx + Dt$

we transform to $x' = 0$. so $Ax + Bt = 0$
We define $v = \frac{x}{t}$

So: $x = vt$

We can plug these in: $Avt + Bt = 0$ $Av + B = 0$ $\frac{A}{B} = -v$

12.3 Sesquilinear forms

12.3.1 Sesquilinear forms

Bilinear form recap

A bilinear form takes two vectors and produces a scalar from the underyling field.

The function is linear in addition in both arguments.

$\phi(au + x, bv + y) = \phi(au, bv) + \phi(au, y) + \phi(x, bv) + \phi(x, y)$

The function is also linear in multiplication in both arguments.

$\phi(au + x, bv + y) = ab\phi(u, v) + a\phi(u, y) + b\phi(x, v) + \phi(x, y)$

They can be represented as:

$\phi(u, v) = v^T Mu$

Sesquilinear forms

Like bilinear forms, sesquilinear are linear in addition:

$\phi(au + x, bv + y) = \phi(au, bv) + \phi(au, y) + \phi(x, bv) + \phi(x, y)$

Sesqulinear forms however are only multiplictively linear in the second argument.

$\phi(au + x, bv + y) = b\phi(au, v) + \phi(au, y) + b\phi(x, v) + \phi(x, y)$

In the first argument they are "twisted"

$\phi(au + x, bv + y) = \bar{a}b\phi(u, v) + \bar{a}\phi(u, y) + b\phi(x, v) + \phi(x, y)$

The real field

For the real field, $\bar{b} = b$ and so the sesqulinear form is the same as the bilinear form.

Representing sesquilinear forms

We can show the sesquilinear form as v^*Mu

Stuff

$f(M) = f([v_1, v_2])$

We introduce e_i, the element vector. This is 0 for all entries except for i where it is 1. Any vector can be shown as a sum of these vectors multiplied by a scalar.

$f(M) = f([\sum_{i=1}^m a_{1i}e_i, \sum_{i=1}^m a_{2i}e_i])$

$f(M) = \sum_{k=1}^m f([a_{1k}e_k, \sum_{i=1}^m a_{2i}e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m f([a_{1k}e_k, a_{2i}e_i])$

Because this in linear in scalars:

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} f([e_k, e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} e_k^* M e_i$

Orthonormal basis and $M = I$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} e_k^* M e_i$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} e_k^* e_i$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} \delta_i^k$

$f(M) = \sum_{i=1}^{m} a_{1i}^* a_{2i}$

12.3.2 Unitary groups $U(n, F)$

Metric preserving transformations for sesquilinear forms

For bilinear forms, the transformations which preserved metrics were:

$P^T = P^{-1}$

For sesquilinear they are different:

$u^* M v$

$(Pu)^* M (Pv)$

$u^* P^* M P v$

So we want the matrices where:

$P^* M P = M$

The unitary group

The unitary group is where $M = I$

$P^* P = I$

$P^* = P^{-1}$

We refer to these using U instead of P.

$U^* = U^{-1}$

Parameters of the unitary group

The unitary group depends on the dimension of the vector space, and the underlying field. So we can have:

- $U(n, R)$; and
- $U(n, C)$.

We generally refer only to the complex

For the $U(n, R)$ we have:

$U^* = U^{-1}$

$U^T = U^{-1}$

This is the condition for the orthogonal group, and so we would instead write $O(n)$.

As a result, $U(n)$ refers to $U(n,C)$.

$U(1)$: The circle group

12.4 Inner products

12.4.1 Symmetric matrices

12.4.2 Hermitian (self-adjoint) matrices

A matrix where $M = M^*$

For matrices over the real numbers, these are the same as symmetric matrices.

Sesqulinear forms on Hermitian matrices

$\phi(u,v) = u^*Mv$

$(u^*Mv)^* = v^*M^*u = v^*Mu$

$\phi(u,v) = \overline{\phi(v,u)}$

The forms on the same vector are always real

$(v^*Mv)^* = v^*M^*v = v^*Mv$

So we have:

$(v^*Mv)^* = v^*Mv$

Which is only satisfied for reals.

If A and B are Hermitian

If A and B are Hermitian, AB is Hermitian if and only if AB commutes.

$(AB)^* = B^*A^* = BA$

If it commutes then

$(AB)^* = AB$

Real eigenvalues

Hermitian matrices have real eigenvalues.

$Hv = \lambda v$

$v^* Hv = \lambda v^* v$

$v^* Hv = \lambda$

Skew-Hermitian matrices

These are also known as anti-Hermitian matrices.

$M^* = -M$

If eigenvalues are different, eigenvectors are orthogonal

12.4.3 Pauli matrices

Pauli matrices are 2×2 matrices which are unitary and hermitian.

That is, $P^* = P^{-1}$.

And $P^* = P$.

The Pauli matrices

The matrices are:

$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

The identity matrix is often considered alongside these as:

$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Pauli matrices are their own inverse

$\sigma_i^2 = \sigma_i \sigma_i$

$\sigma_i^2 = \sigma_i \sigma_i^*$

$\sigma_i^2 = \sigma_i \sigma_i^{-1}$

$\sigma_i^2 = I$

Determinants and trace of Pauli matrices

$\det \sigma_i = -1$

$Tr(\sigma_i) = 0$

As the sum of eigenvalues is the trace, and the product is the determinant, the eigenvalues are 1 and -1.

12.4.4 Positive-definite matrices

The matrix M is positive definite if for all non-zero vectors the scalar is positive.

$v^T M v$

We know that the outcome is a scalar, so:

$v^T M v = (v^T M v)^T$

$v^T M v = v^T M^T v$

$v^T (M - M^T) v = 0$

12.4.5 Inner products

An inner product is a sesquilinear form with a positive-definite Hermitian matrix.

$\langle u, v \rangle = u^* H v$

If we are using the real field this is the same as:

$\langle u, v \rangle = u^T H v$

Where H is now a symmetric real matrix.

Same

$\langle v, v\rangle = v^*Hv$

Always positive and real.

Properties

$\langle u, v\rangle\langle v, u\rangle = |\langle u, v\rangle|^2$

12.4.6 Cauchy-Schwarz inequality

This states that:

$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\dot{\langle} v, v\rangle$

Consider the vectors u and v. We construct a third vector $u - \lambda v$. We know the length of any vector is non-negative. $0 \leq \langle u - \lambda v, u - \lambda v\rangle$

$0 \leq \langle u, u\rangle + \langle u, -\lambda v\rangle + \langle -\lambda v, u\rangle + \langle -\lambda v, -\lambda v\rangle$

$0 \leq \langle u, u\rangle - \bar{\lambda}\langle u, v\rangle - \lambda\langle v, u\rangle + \lambda\bar{\lambda}\langle v, v\rangle$

We now look for a value of λ to simplify this equation.

$\lambda = \frac{\langle u, v\rangle}{\langle v, v\rangle}$

$0 \leq \langle u, u\rangle - \frac{\langle v, u\rangle\langle u, v\rangle}{\langle v, v\rangle} - \frac{\langle u, v\rangle\langle v, u\rangle}{\langle v, v\rangle} + \frac{\langle u, v\rangle}{\langle v, v\rangle}\frac{\langle v, u\rangle}{\langle v, v\rangle}\langle v, v\rangle$

$0 \leq \langle u, u\rangle - \frac{|\langle u, v\rangle|^2}{\langle v, v\rangle}$

$|\langle u, v\rangle|^2 \geq \langle u, u\rangle\langle v, v\rangle$

12.4.7 The orthogonal projection

in inner product space, orthogonal projection

$p_u v = \frac{< u, v >}{v, v} v$

we then know that $o = v - p_u v$ is orthogonal to u.

12.4.8 Orthogonal set / Orthogonal basis

if a set of vectors are all orthogonal, they form an orthogonal set if the set spans the vector space, it is an orthogonal basis.

12.4.9 The Gram-Schmidt process

can we form an orthogonal basis from a non-orthogonal basis? yes, using gram schmidt

we have x_1, x_2 x_3 etc we want to make v_1, v_2 etc orthognal

$v_1 = x_1$ $v_2 = x_2 - p_{x_2} v_1$ $v_3 = x_3 - p_{x_3} v_1 - p_{x_3} v_2 v$

12.5 Multilinear forms and determinants

12.5.1 Multilinear forms

12.5.2 Determinants

From invertible matrix section in endo

A matrix can only be inverted if it can be created from a combination of elementary row operations.

How can we identify if a matrix is invertible? We want to create a scalar from the matrix which tells us if this possible. We can this scalar the determinant.

For a matrix A we label the determinant $|A|$, or $\det A$

We propose $|A| = 0$ when the matrix is not invertible.

So how can we identify the function we need to undertake on the matrix?

New 1

We know that linear dependence results in determinants of 0.

We can model this as a function on the columns of the matrix.

$\det M = \det([M_1, ..., M_n)$

If there is linear depednence, for example if two columns are the same then:

$\det([M_1, ..., M_i, ..., M_i, ..., M_n]) = 0$

Similarly, if there is a column of 0 then the determinant is 0.

$\det([M_1, ..., 0, ..., M_n]) = 0$

New 2

Show linear in addition

How can we identify the determinant of less simple matrices? We can use the multilinear form.

$\sum c_i \mathbf{M}_i = \mathbf{0}$

Where $\mathbf{c} \neq \mathbf{0}$

Or:

$M\mathbf{c} = \mathbf{0}$

Rule 1: Columns of matrices can be the input to a multilinear form

A matrix can be shown in terms of its columns. $A = [v_1, ..., v_n]$

$\det A = \det[v_1, ..., v_n]$

$\det A = \sum_{k_1=1}^{m} ... \sum_{k_n=1} \prod_{i=1}^{m} a_{ik_i} \det([e_{k_1}, ..., e_{k_n}])$

Multiplying a matrix by a constant multiplies the determinant by the same amount

If a whole row or columns is 0 then:

$\det A = \det[v_1, ..., v_i, ..., v_n]$

$\det A' = \det[v_1, ..., cv_i, ..., v_n]$

$\det A = \det[v_1, ..., v_i, ..., v_n]$

$\det A' = \det[v_1, ..., cv_i, ..., v_n]$

$\det A' = c \det[v_1, ..., v_i, ..., v_n]$

$\det A' = c \det A$

As a result, multiplying a column by 0 makes the determinant 0.

A matrix with a column of 0 therefore has determinant 0

Rule 2: A matrix with equal columns has a determinant of 0.

$A = [a_1, ..., a_i, ..., a_i, ..., a_n]$

$D(A) = D([a_1, ..., a_i, ..., a_i, ..., a_n])$

We know from Result 3 that swapping columns reverses the sign. Reversing columns results in the same matrix, so the determinant must be unchanged.

$D(A) = -D(A)$

$D(A) = 0$

Linear dependence

If a column is a linear combination of other columns, then the matrix cannot be inverted.

$A = [a_1, ..., \sum_{j \neq i}^{n} c_j a_j, ..., a_n]$

$\det A = \det([v_1, ..., \sum_{j \neq i}^{n} c_j v_j, ..., v_n])$

$\det A = \sum_{j \neq i}^{n} c_j \det([v_1, ..., v_j, ..., v_n])$

$\det A = \sum_{j \neq i}^{n} c_j \det([v_1, ..., v_j, , ..., v_j, ..., v_n])$

As there is a repeating vector:

$\det A = 0$

Swapping columns multiplies the determinant by -1

$A = [v_1, ..., v_i + v_j, ..., v_i + v_j, ..., v_n]$

We know.

$\det A = 0$

$\det A = \det([a_1, ..., a_i, ..., a_i, ..., a_n]) + \det([a_1, ..., a_i, ..., a_j, ..., a_n]) + \det([a_1, ..., a_j, ..., a_i, ..., a_n]) + \det([a_1, ..., a_j, ..., a_j, ..., a_n])$

So:

$\det([a_1, ..., a_i, ..., a_i, ..., a_n]) + \det([a_1, ..., a_i, ..., a_j, ..., a_n]) + \det([a_1, ..., a_j, ..., a_i, ..., a_n]) + \det([a_1, ..., a_j, ..., a_j, ..., a_n]) = 0$

As 2 of these have equal columns these are equal to 0.

$\det([a_1, ..., a_i, ..., a_j, ..., a_n]) + \det([a_1, ..., a_j, ..., a_i, ..., a_n]) = 0$

$\det([a_1, ..., a_i, ..., a_j, ..., a_n]) = -\det([a_1, ..., a_j, ..., a_i, ..., a_n])$

Calculating the determinant

We have

$\det A = \sum_{k_1=1}^{m} ... \sum_{k_n=1} \prod_{i=1}^{m} a_{ik_i} \det([e_{k_1}, ..., e_{k_n}])$

So what is the value of the determinant here?

We know that the determinant of the identity matrix is 1.

We know that the determinant of a matrix with identical columns is 0.

We know that swapping columns multiplies the determinant by -1.

Therefore the determinants where the values of k are not all unique are 0.

The determinants of the others are either -1 or 1 depending on how many swaps are required to restore to the identity matrix.

This is also shown as the Leibni formula.

$\det A = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$

12.5.3 Properties of determinants

Identity

$\det I = 1$

Multiplication

$\det(AB) = \det A \det B$

Inverse

$\det(M^{-1}) = \frac{1}{\det M}$

We know this because:

$\det(MM^{-1}) = \det I = 1$

$\det M \det M^{-1} = 1$

$\det(M^{-1}) = \frac{1}{\det M}$

Complex cojugate

$\det(M^*) = \overline{\det M}$

Transpose

$\det(M^T) = \det M$

Addition

$\det(A + B) = \det A + \det B$

Scalar multiplication

$\det cM = c^n \det M$

Determinants and eigenvalues

The determinant is equal to the product of the eigenvalues.

12.5.4 Determinants of 2x2 and 3x3 matrices

The determinant of a 2x2 matrix

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|M| = ad - bc$

The determinant of a 3x3 matrix

$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$|M| = aei + bfg + cdh - ceg - dbi - afh$

12.6 Special groups

12.6.1 Special orthogonal groups $SO(n, F)$

The special orthogonal group, $SO(n, F)$, is the subgroup of the orthogonal group where $|M| = 1$.

As a result it includes only the rotation operators, not the flip operators.

$SO(3)$ is rotations in 3d space.

$SO(2)$ is rotations in 2d space.

Determinant of the orthogonal group

The orthogonal group has determinants of -1 or 1.

$O^T = O^{-1}$

$\det(O^T) = \det(O^{-1})$

$\det O = \frac{1}{\det O}$

$\det O = \pm 1$

12.6.2 Special unitary groups $SU(n, F)$

The special unitary group, $SU(n, F)$, is the subgroup of $U(n, F)$ where the determinants are 1.

That is, $|M| = 1$

The determinant of unitary matrices

The determinant of the unitary matrices is:

$\det U^* = \det U^{-1}$

$(\det U)^* = \frac{1}{\det U}$

$(\det U)^* \det U = 1$

$|| \det U|| = 1$

12.6.3 Special lnear groups $SL(n, F)$

The special linear group, $SL(n, F)$, is the subgroup of $GL(n, F)$ where the determinants are 1.

That is, $|M| = 1$

These are endomorphisms, not forms.

12.7 Sort

12.7.1 Normal matrices

$M^* M = M M^*$

All symmetrix matrices are normal

All hermetitian matrices (inc subset symmetric) are normal

Normal matrix never defective

Chapter 13

Linear maps

13.1 Homomorphisms of vector spaces

13.1.1 Linear maps

Homomorphisms between vector spaces

Homomorphisms map between algebras, preserving the underlying structure.

A homomorphism vetween vector space V and vector space W can be described as:

$\hom(V, W)$

Homomorphism between vector spaces must preserve the group-like structure of the vector space.

$f(u+v) = f(u) + f(v)$

The homomorphism must also preserve scalar multiplication.

$f(\alpha v) = \alpha f(v)$

A linear map (or function) is a map from one input to an output which preserves addition and scalar multiplication.

That is if function f is linear then:

$f(aM + bN) = af(M) + bf(N)$

Alternative names for homomorphisms

Vector spaces homomorphisms are also called linear maps or linear functions.

13.1.2 Homomorphisms form a vector space

If we can can show that scalars can act on morphisms, then we can shwn that morphisms on a vector space are themselves a vector space.

Scalars can act on morphisms, and so morphisms of vector spaces are themselves vector spaces.

Dimensions of homomorphisms

We can identify the dimensionality of this new vector space from the dimensions of the original vector spaces.

$\dim(\hom(V, W)) = \dim V \dim W$

13.1.3 The pseudo-inverse

The definition of the inverse is that:

$MM^{-1} = I$

$M^{-1}M = I$

We also have:

$MM^{-1}M = M$

$M^{-1}MM^{-1} = M^{-1}$

The inverse of a homomorphism

Generally we don't have inverses of homomorphisms as the number of dimensions are different.

We can, however, find a matrix M^+ which satisfies:

$MM^+M = M$

$M^+MM^+ = M^+$

This is the pseudo-inverse.

13.1.4 Linear and affine functions

Linear maps

Linear maps can be written as:

$v = Mu$

These go through the origin. That is, if $u = 0$ then $v = 0$.

Affine function

Affine functions are more general than linear maps. They can be written as:

$v = Mu + c$

Where c is a vector in the same space as v.

Affine functions where $c \neq 0$ are not linear maps. They are not homomorphisms which preserve the structure of the vector space.

If we multiply u by a scalar s, then v will not increase by the same proportion.

13.1.5 Singular value decomposition

The singular value decomposition of $m \times n$ matrix M is:

$M = U\Sigma V^*$

Where:

- U is a unitary matrix ($m \times m$)
- Σ is a diagonal matrix with non-negative real numbers ($m \times n$)
- V is a unitary matrix ($n \times n$)

Σ is unique. U and V are not.

Properties

$M^*M = U\Sigma^2U^*$

$(M^*M)^{-1} = V\Sigma^{-2}V^*$

Calculating the SVD

The SVD is generally calculated iteratively.

13.1.6 Identity matrix and the Kronecker delta

The Kronecker delta

The Kronecker delta is defined as:

p$\delta_{ij} = 0$ where $i \neq j$

$\delta_{ij} = 1$ where $i = j$

We can use this to define matrices. For example for the identity matrix:

$I_{ij} = \delta_{ij}$

Identity matrix

A square matrix where every element is 0 except where $i = j$. There is one for each square matrix.

$$I = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix}$$

Chapter 14

Lie algebra

14.1 Cross products

14.1.1 The cross product

$v \times u$

Cross product is a bilinear map

This is a bilinear map from two vectors in $\mathbb{R}^3$ to another vector in the same space.

$V \times V \to V$

Calculating the cross product

This is calcualted by:

$u \times v = ||u||||v|| \sin(\theta)n$

The resulting vector is perpendicular to both input vectors.

14.2 Lie groups

14.2.1 Lie groups

14.3 Lie algebra

14.3.1 Lie algebra

Lie groups have symmetries. We can consider only the infintesimal symmetries.

For example the unit circle has many symmetries, but we can consider only those which rotate infintesimally.

Example

Take a continous group, such as $U(1)$. Its Lie algebra is all matrices such that their exponential is in the Lie group.

$\mathfrak{u}(1) = \{X \in \mathbb{C}^{1\times 1} | e^{tX} \in U(1) \forall t \in \mathbb{R}\}$

This is satisfied by the matrices where $M = -M^*$. Note that this means the diagonals are all 0.

Scale of specific Lie algebra matrices doesn't matter

Because of t.

Commutation of Lie group algebra

Consider two members of the Lie algebra: A and B. The commutator is:

A.

The corresponding Lie group member is:

$e^{t(A+B)} = e^{tA}e^{tB}$

While the Lie group multiplication may not commute, the corresponding addition of the Lie algebra does.

14.3.2 The Lie bracket

We can define the Lie bracket from the ring commutator.

We use the Lie bracket, rather than multiplication, as the operator over a field homomorphism.

$[A, B]$

This generates another element in the algebra.

This satisifies:

- Bilinearity: $[xA + yB, C] = x[A, C] + y[B, C]$
- Alternativity: $[A, A] = 0$
- Jacobi identity: $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$
- Anticommutivity: $[A, B] = -[B, A]$

One option for the Lie bracket is the ring commutor. So that:

$[A, B] = AB - BA$

14.3.3 Commutation of Lie groups

We can measure commutation of Lie groups using:

$ABA^{-1}B^{-1}$

If the group commutes then:

$ABA^{-1}B^{-1} = BA^{-1}B^{-1} = I$

Commutation of Lie algebra: COMPLETE THIS

This corresponds to $[A, B] = AB - BA$ in the underlying lie algebra, if we expand.

$A = e^{ta}$

$B = e^{tb}$

$ABA^{-1}B^{-1} = e$

14.4 Lie algebra of specific Lie groups

14.4.1 Lie algebra of $O(n)$

$O(n)$ forms a Lie group

Lie algebra of $O(n)$

The Lie algebra of $O(n)$ is defined as:

$\mathfrak{o}(n) = \{X \in \mathbb{R}^{n\times n} | e^{tX} \in O(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-symmetric matrices where $M = -M^T$. Note that this means the diagonals are all 0.

14.4.2 Lie algebra of $U(n)$

$U(n)$ forms a Lie group

Lie algebra of $U(n)$

The Lie algebra of $U(n)$ is defined as:

$\mathfrak{u}(n) = \{X \in \mathbb{C}^{n\times n} | e^{tX} \in U(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-Hermitian matrices where $M = -M^*$. Note that this means the diagonals are all 0 or pure imaginary.

14.4.3 Lie algebra of $SO(n)$

$SO(n)$ forms a Lie group

Lie algebra of $SO(n)$

The Lie algebra of $SO(n)$ is defined as:

$\mathfrak{so}(n) = \{X \in \mathbb{R}^{n\times n} | e^{tX} \in SO(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-symmetric matrices where $M = -M^T$. Note that this means the diagonals are all 0.

14.4.4 Lie algebra of $SU(n)$

$SU(n)$ forms a Lie group

Lie algebra of $SU(n)$

The Lie algebra of $SU(n)$ is defined as:

$\mathfrak{su}(n) = \{X \in \mathbb{C}^{n\times n} | e^{tX} \in SU(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-Hermitian matrices where $M = -M^*$ and the trace is 0. Note that this means the diagonals are all 0 or pure imaginary.

14.5 Hypercomplex numbers

14.5.1 Hypercomplex numbers

14.5.2 Quaternions

14.5.3 Clifford algebra

14.6 Sort

14.6.1 Projective line in the field

Chapter 15

Exterior algebra

15.1 Exterior algebra

15.1.1 The exterior (wedge) product

The exterior product of two vectors is:

$u \wedge v$

15.1.2 The exterior product is anticommutative

This is anticommutative (alternating).

$u \wedge v = -v \wedge u$

This implies that:

$u \wedge u = 0$

15.1.3 The exterior product is distributive

$(a + b) \wedge (c + d) = (a \wedge c) + (a \wedge d) + (b \wedge c) + (b \wedge d)$

15.1.4 Expanding the exterior product of two vectors

We can look at the exterior product in component-basis terms.

Consider 2-dimenional vector space with the following vectors:

$u = ae_1 + be_2$

$v = ce_1 + de_2$

The exterior product is:

$u \wedge v = (ae_1 + be_2) \wedge (ce_1 + de_2)$

$u \wedge v = (ae_1 \wedge ce_1) + (ae_1 \wedge de_2) + (be_2 \wedge ce_1) + (be_2 \wedge de_2)$

$u \wedge v = ac(e_1 \wedge e_1) + ad(e_1 \wedge e_2) + bc(e_2 \wedge e_1) + bd(e_2 \wedge e_2)$

$u \wedge v = ad(e_1 \wedge e_2) - bc(e_1 \wedge e_2)$

$u \wedge v = (ad - bc)(e_1 \wedge e_2)$

15.1.5 Exterior (Grassman) algebra

The exterior algebra is the algebra generated by the wedge product.

The term $u \wedge v$ can be interpreted as the area covered by the parallelogram generated by u and v.

As $a\mathbf{u} \wedge b\mathbf{v} = ab\mathbf{u} \wedge \mathbf{v}$, we can see that scaling the length of one of the vectors by a scalar, we also increase the exterior product by the same scalar.

15.1.6 Orientation

We can describe the exterior product of two vectors as $\mathbf{u} \wedge \mathbf{v}$ or $\mathbf{v} \wedge \mathbf{u}$.

15.1.7 Bivectors

15.1.8 Trivectors

Chapter 16

Tensors

16.1 Element-wise notation

16.1.1 Einstein summation convention

A vector can be written as a sum of its components.

$v = \sum_i e_i v^i$

The Einstein summation convention is to remove the $\sum_i$ symbols where they are implicit.

For example we would instead write the vector as:

$v = e_i v^i$

Adding vectors

$v + w = (\sum_i e_i v^i) + (\sum_i f_i w^i)$

$v + w = \sum_i (e_i v^i + f_i w^i)$

$v + w = e_i v^i + f_i w^i$

If the bases are the same then:

$v + w = e_i (v^i + w^i)$

Scalar multiplication

$cv = c \sum_i e_i v^i$

$cv = \sum_i ce_i v^i$

$cv = ce_i v^i$

Matrix multiplication

$AB_{ik} = \sum_j A_{ij} B_{jk}$

$AB_{ik} = A_{ij} B_{jk}$

Inner products

$\langle v, w\rangle = \langle \sum_i e_i v^i, \sum_j f_j w^j\rangle$

$\langle v, w\rangle = \sum_i v^i \langle e_i, \sum_j f_i w^j\rangle$

$\langle v, w\rangle = \sum_i \sum_j v^i \overline{w^j} \langle e_i, f_j\rangle$

If the two bases are the same then:

$\langle v, w\rangle = \sum_i \sum_j v^i \overline{w^j} \langle e_i, e_j\rangle$

We can define the metric as:

$g_{ij} := \langle e_i, e_j\rangle$

$\langle v, w\rangle = v^i \overline{w^j} g_{ij}$

16.1.2 Covariant and contravariant bases

In element form we write a vector as:

$v = e_i v^i$

The indices are raised and lowered to reflect whether the value is covariant or contravariant.

v^i is contravariant. If the basis moves one way, it moves the other.

e_i is covariant. If the basis moves, it moves with it.

16.2 Tensor product

16.2.1 Tensor product

We have spaces V and W over field F. If we have a linear operation which takes a vector from each space and returns a scalar from the underlying field, it is an element of the tensor product of the two spaces.

For example if we have two vectors:

$v = e_i v^i$

$w = e_j w^j$

A tensor product would take these and return a scalar.

There are three types of tensor products:

- Both are from the vector space
- $T_{ij} v^i w^j$
- $T_{ij} \in V \otimes W$
- Both are from the dual space
- $T^{ij} v_i w_j$
- $T_{ij} \in V^* \otimes W^*$
- One is from each space
- $T_i^j v^i w_j$
- $T_{ij} \in V \otimes W^*$

As a vector space, we can add together tensor products, and do scalar multiplication.

Basis of a tensor product

Not all elements spanned by a basis of a tensor product are themselves tensor products.

Eigenvalues and Eigenvectors of a tensor product

Homomorphisms

We can define homomorphisms in terms of tensor products.

$Hom(V) = V \otimes V^*$

T_j^i

We use the dual space for the second argument. This is because it ensures that changes to the bases do not affect the maps.

$w^j = T_i^j v^i$

16.2.2 Raising and lowering indices

We showed that the inner product between two vectors with the same basis can be written as:

$\langle v, w\rangle = \langle \sum_i e_i v^i, \sum_j f_j w^j \rangle$

$\langle v, w\rangle = v^i \overline{w^j} \langle e_i, e_j \rangle$

Defining the metric as:

$g_{ij} := \langle e_i, e_j \rangle$

$\langle v, w\rangle = v^i \overline{w^j} g_{ij}$

Metric inverse

We can use this to define the inverse of the metric.

$g^{ij} := (g_{ij})^{-1}$

We can use this to raise and lower vectors.

$v_i := v^j g_{ij}$

Raising and lowering indices of tensors

If we have tensor:

T_{ij}

We can define:

$T_i^k = T_{ij} g^{jk}$

$T^{il} = T_{ij} g^{jk} g^{kl}$

Tensor contraction

If we have:

$T_{ij} x^j$

We can contract it to:

$T_{ij} x^j = v_i$

Similarly we can have:

$T^{ij} x_j = v^i$

16.2.3 Kronecker delta

Consider matrix multiplication AI.

We have:

$AI_{ik} = A_{ij}I_{jk}$

We write this instead as:

$AI_{ik} = A_{ij}\delta_{jk}$

Where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$.

16.2.4 Tensors form a vector space

Recap

Tensors form a vector space

Dimension of a tensor

Basis of a tensor

16.3 Tensors

16.3.1 Tensor valence

16.3.2 Tensor inverses

For second order tensors we have:

- T^i_j
- T_{ij}
- T^{ij}

For each of these we can define an inverse:

- $T^j_i U^k_j = \delta^k_i$
- $T_{ij}U^{jk} = \delta^k_i$
- $T^{ij}U_{jk} = \delta^k_i$

Notation for inverses

If we have $T_{ij}U^{jk} = \delta_i^k$, we can instead write:

$T_{ij}T^{jk} = \delta_i^k$

16.3.3 Tensor contraction

We have a vector $v \in V$ and $w \in V^*$.

$\mathbf{v} = \sum_i v^i \mathbf{e}_i$

$\mathbf{w} = \sum_i w_i \mathbf{f}^i$

$\mathbf{wv} = [\sum_i v^i \mathbf{e}_i][\sum_i w_i \mathbf{f}^i]$

$\mathbf{wv} = \sum_i \sum_j [v^i \mathbf{e}_i][w_j \mathbf{f}^j]$

$\mathbf{wv} = \sum_i \sum_j v^i w_j \mathbf{e}_i \mathbf{f}^j$

We use the dual basis so:

$\mathbf{wv} = \sum_i \sum_j v^i w_j \mathbf{e}_i \mathbf{e}^j$

$\mathbf{wv} = \sum_i \sum_j v^i w_j \delta_i^j$

We can see that this value is unchanged when there is a change in basis.

What if these were both from V?

$\mathbf{v} = \sum_i v^i \mathbf{e}_i$

$\mathbf{w} = \sum_i w^i \mathbf{e}_i$

$\mathbf{wv} = [\sum_i v^i \mathbf{e}_i][\sum_i w^i \mathbf{e}_i]$

$\mathbf{wv} = \sum_i \sum_j v^i w^j \mathbf{e}_i \mathbf{e}_i$

This term is dependent on the basis, and so we do not contract.

So if we have $v_i w^i$, we can contract, because the result (calculated from the components) does not depend on the basis.

But if we have $v_i w_i$, the result (calcualted from the components) will change depending on the choice of basis.

We define a new object

$c = \sum_i w^i v_i$

This new term, c, does not depend on i, and so we have contracted the index.

16.3.4 Symmetric and antisymmetric tensors

Consider a tensor, e g T_{abc}.

In general, this is not symmetric, that is:

$T_{abc} \neq T_{bac}$

Symmetric part of a tensor

We can write the symmetric part of this with regard to a and b.

$T_{(ab)c} = \frac{1}{2}(T_{abc} + T_{bac})$

Clearly, $T_{(ab)c} = T_{(ba)c}$

Antisymmetric part of a tensor

We can also have an an antisymmetric part with regard to a and b.

$T_{[ab]c} = \frac{1}{2}(T_{abc} - T_{bac})$

Clearly, $T_{[ab]c} = -T_{[ba]c}$

Tensors as sums of their symmetric and antisymmetric parts

$T_{(ab)c} + T_{[ab]c} = \frac{1}{2}(T_{abc} + T_{bac}) + \frac{1}{2}(T_{abc} - T_{bac})$

$T_{(ab)c} + T_{[ab]c} = T_{abc}$

16.4 Higher-order tensors

16.4.1 Higher-order tensors

We can create higher order tensors products. For example

$V \otimes V \otimes V \otimes V^* \otimes V^*$

We write elements of these as:

$T^{i_1,\dots,i_p}_{j_1,\dots,j_q}$

We can map from matrix to matrix etc higher dimensional

Matrix has A: a_{ij}.

Tensor can have T: t_{ijk} for example

0 rank tensor: scalar

1 rank tensor: vector

2 rank tensor: matrix

page on covariance and contravariance and type (p, q)

16.5 Sort

16.5.1 Outer product

The outer product is a bilinear map

This is a bilinear map from two vectors from the same vector space to another vector space.

$V \times V \to V$

Calculating the outer product

$u \otimes v = w$

$w_{ij} = u_i v_j$

The dimensions of the tensor outer product

$\dim(V \otimes W) = \dim V \times \dim W$

Outer product on the complex numbers

Relation between the dot product and outer product

The dot product in the trace of the outer product.

16.5.2 Kronecker product

The Kronecker product takes the concept of the outer product and applies to to matrices.

We can essentially repace every element in the matrix on the left with the element multiplied by the entire matrix on the right.

Like outer products, Kronecker products are written as:

$u \otimes v = w$

16.5.3 Dot product

Dot product is a bilinear form

This is a bilinear form, a mapping from two vectors in the same vector space to the underlying field.

$V \times V \to F$

Calculating the dot product

This is calculated by multiplying each matching element, and summing the results.

$u \cdot v = \sum_{i=1}^{n} u_i v_i$

Dot product on the complex numbers

Properties don't hold. Can get zero vectors from non-zero inputs. Get complex numbers from dot product on itself.

Inner products better deal with complex number fields. However they are not bilinear maps.

16.5.4 Homomorphism as a tensor product

16.5.5 Tensors

A tensor is an element of a tensor product.

Chapter 17

Infinite-dimensional vector spaces

17.1 Real functions as infinite-dimensional vectors

17.1.1 Real functions are vectors

The real function space is a vector space because it is linear in multiplication and addition.

$g(x) = cf(x)$

$h(x) = f(x) + k(x)$

17.2 Endomorphisms of infinite-dimensional vector spaces

17.2.1 Endomorphisms on real functions

We start with our vector $f(x)$.

$h(x) = f(x)g(x)$

The equivalent of the identity matrix is where $g(x) = 1$.

These are similar to endomorphisms where all off diagonal elements are 0.

Differentiation

$h(x) = \frac{\delta}{\delta x} f(x)$

Integration

$h(x) = \int_{-\infty}^{x} f(z) dz$

17.2.2 Examples of linear operators on real functions

For a function v we can define operators Ov.

Here we consider some examples and their properties.

Real multiplication

$Rv = rf(x)$

This operator is hermitian. This is equivalent to a finite operator of the form rI.

Multiplication by underlying real number

$Xv = xf(x)$

This operator is hermitian. This is equivalent to a finite operator of the form $M_{ii} = i$ and $M_{ij} = 0$.

Differentiation

$Dv = \frac{\delta}{\delta x} f(x)$

While this operator is not hermitian, the following is:

$-iDv = \frac{\delta}{\delta x}[-if(x)]$

17.3 Eigenvalues and eigenvectors of infinite-dimensional vectors

17.3.1 Spectral theorem for infinite-dimensional vector spaces

17.4 Forms on infinite-dimensional vector spaces

17.4.1 Forms on real functions

A form takes two vectors and produces a scalar.

Integration as a form

We can use integration to get a bilinear form.

$\int f(x)g(x)dx$

If we instead want a sesquilinear form we can instead use:

$\int f(\bar{x})g(x)dx$

17.4.2 Functionals

Functionals map functions to scalars. They are the 1-forms of infinite-dimensional vector spaces.

If we have a function f, we can write functional $J[f]$.

More

We can define neighbourhoods around a function f. For example, taking y to be f with infintesimal changes. to each of the values.

The difference between the functional at both points is

$\delta J = J[y] - J[f]$

Extrema

If

$\delta J = J[y] - J[f]$

is the same sign for all y around f, then J has an extremum at f.

Functional derivatives

17.4.3 Hilbert space

A complete space with an inner product. That is, a Banach space where the norm is derived from an inner product.

17.5 Calculus of variations

17.5.1 Calculus of variations

17.5.2 Functional integration

Integrate over possible functions?

17.6 Sort

17.6.1 Banach space

A complete normed vector space

17.6.2 Wave functions

For a vector in hermitian basis, for each eigenvector we have component. wave function is function on ith component.

17.7 Other

17.7.1 Dirac delta

Kronecker delta

The function is: δ_{ij}

If $i = j$ this is 1. Otherwise it is 0.

We introduced this in linear algebra.

Dirac delta

The Dirac delta replaces the Kronecker delta for continuous functions.

That is, we want:

- $\delta(x \neq 0) = 0$
- $\delta(0) = +\infty$
- $\int_{-\infty}^{\infty} \delta(x) dx = 1$

Part IV

Manifolds

Chapter 18

Exterior calculus

18.1 Introduction

18.1.1 Differential forms

18.1.2 Exterior derivative

18.1.3 The fundamental theorem of external calculus

Chapter 19

Calculus of variations

19.1 sort

19.1.1 Frchet derivative

19.1.2 Gteaux derivative

19.1.3 Euler-Lagrange equations

Chapter 20

Topology of finite sets

20.1 Nearness functions

20.1.1 Topologies

20.1.2 Topologies on sets

T is a topology on set X if:

- $X \in T$
- $\varnothing \in T$
- Unions of T are in T
- Intersections of T are in T

20.1.3 Examples of topologies: The trivial topology

The trivial topology contains only the underlying set and the empty set.

20.1.4 Examples of topologies: The discrete toplogy

The discrete toplogy contains all subsets of the underlying set (is this the power set?)

20.2 Neighbourhoods

20.2.1 Neighbourhood topology

We have a set X.

For each element $x \in X$, there is a non-empty set of neighbourhoods $N \in \mathbf{N}(x)$ where $x \in N \subseteq X$ such that:

- If N is a subset of M, M is a neighbourhood.
- The intersection of two neighbourhoods of x is a neighbourhood of x.
- N is a neighbourhood for each point in some $M \subseteq N$

20.2.2 Topological distinguishability

If two points have the same neighbourhoods then they are topologically indistinguishable.

For example in the trivial topology, all points are topologically indistinguishable.

20.2.3 Open sets

U is an open set if it is a neighbourhood for all its points.

20.3 Open and closed sets

20.3.1 Limit points and closure

Limit points

A point x in the topological set X is a limit point for $S \subset X$ if every neighbourhood of x contains another point in S.

For example -1 is a limit point for the real numbers where S is $[0, 1]$ (or $(0, 1)$.

Closure

The closure of a subset of a topological space is the subset itself along with all limit points.

So the closure of $|x| < 1$ includes -1 and 1.

20.3.2 Boundries and interiors

The boundry of the subset S of a topology is the intersection with the closure of S with the closure of the complement of S.

So the boundry of both $(0,1)$ and $[0,1]$ are 0 and 1.

The interior of S is S without the boundry.

So the interior of $(0,1)$ and $[0,1]$ are both $(0,1)$.

20.3.3 Closed sets

The complement of any open set is a closed set.

A set can be open, closed, both or neither.

20.4 Compactness

20.4.1 Covers

A space X is covered by a set of subsets of X, C, if the union of C is X.

20.4.2 Subcover

A subset of C which still covers X is a a subcover.

20.4.3 Open cover

C is an open cover if each member is an open set.

20.4.4 Universal cover

20.4.5 Bases of topologies

Subset B of topology T is a base for T if all elements of T are unions of members of B.

Second-countable space

If B is finite then the toplogy is a second-countable space.

20.5 Separation

20.5.1 Connected and separated sets

Two subsets of X in topological space T are separated if each subset is disjoint from the other's closure.

So $[-1, 0)$ and $(0, 1)$ are separated.

$[-1, 0]$ and $(0, 1)$ are not separated.

Sets which are not separated are connected.

20.6 Cartesian products

20.6.1 Box topology

20.6.2 Product topology

20.7 Creating topologies from sets

20.7.1 The trivial topology

A topology which contains just X and $\varnothing$ is the trivial topology.

20.7.2 Discrete topology

20.8 Taxonomy of spaces

20.8.1 Lindelf space

In a Lindelf space all open covers have countable subcovers.

This is weaker than compactness, which requires that every open cover has a finite subcover.

20.8.2 Kolmogorov space

In a Kolmogorov (or T_0) space, for every pair of points there is a neighbourhood containing one but not the other.

20.9 Local properties

20.9.1 Local properties

Locally, a topology may have properties which are not present globally.

20.9.2 Locally compact spaces

20.9.3 Locally connected spaces

20.10 TO INF

20.10.1 Hausdorf space

In a Hausdorf (or T_2) space, any two different points have neighbourhoods which are disjoint.

20.10.2 Compact spaces

A space X is compact if each open cover has a finite subcover.

If we can define a cover which does not have a finite subcover, then the space is not compact.

For example an infinite cover could be tend towards $(0,1)$, eg as $\frac{1}{n}, 1-\frac{1}{n}$

This covers $(0,1)$, but there is no finite subcover. As a result $(0,1)$ is not compact.

Chapter 21

Topological manifolds

21.1 Introduction

21.1.1 Manifolds, charts and atlases

A manifold is a set of points and associated charts.

A chart is a mapping from each point in a subset of the manifold to a point in a vector space.

These charts are invertible. If we are given coordinates, we can identify the point in the manifold it comes from.

For each point we have a topological neighbourhood. For each point in the neighbourhood, we can map to an element in the tangent space.

Example: The sphere

We can map a hemisphere to a subset of R^2. Given a point in R^2 we can identify a specific point on the hemisphere, and given a s specific point on the hemisphere we can identify a point in R^2.

Universal charts

If the vector space is flat and non-repeating, then a single chart can be used to map the whole manifold.

Atlases

If we have a collection of charts which covers each point needs to be covered at least once, we have an atlas. Each chart needs to be to the same dimensional vector space.

21.1.2 Transition maps

Where two charts overlap we can express the points where the charts overlap as two different coordinates.

We can express the mapping from these coordinates as a function. This is a transition map.

Overlapping charts

If two charts cover some of the same points on a manifold then we can define a function for those points where we move from one vector to another.

We can represent moving between charts as:

ab^{-1}

21.1.3 Mapping 2D manifolds to Riemann surfaces

Needs to be orientable and metricisable.

21.1.4 Connections of topological manifolds

Connected vs path-connected topological manifods.

21.2 Dimension theory

21.2.1 Refinement

21.2.2 Ply (order) of a cover

21.2.3 Small inductive dimension

21.2.4 Large inductive dimension

21.2.5 Lebesgue covering dimension

21.3 Paths

21.3.1 Paths and loops

Paths

We have the set X. We define a mapping $[0,1] \to X$

If a path exists between any two points, then the space is path-connected.

Loops

This is a path which ends on itself.

If $f(0) = f(1)$ then it is a loop.

21.3.2 Holes and genuses

Holes

Genes

The genus of a topology is the number of holes in the topology.

21.3.3 Path-connect spaces

21.4 Simply-connected 2D manifolds

21.4.1 Elliptic (Riemann sphere)

21.4.2 Parabolic (complex plane)

21.4.3 Hyperbolic (open disk)

21.5 Not simply-connected 2D manifolds

21.5.1 Torus

21.5.2 Hyper-elliptic curves

21.6 Functions between topologies

21.6.1 Functions between topologies

We can define a function from topology to another.

$f(X) = Y$

Continuous functions between topologies

If $f(X)$ is continous, then we have a continous function between topologies.

Inverse functions between topologies

If $f(X)$ is invertible then there is a inverse mapping.

21.6.2 Homotopy

21.6.3 Homeomorphisms

If there is a mapping which is invertible and continuous, it is a homeomorphism.

21.7 Fibre bundles

21.7.1 Vector bundles

A vector bundle consists of a base manifold (a base space), and a real vector space at each point in the base manifold.

Example

For example we can have a base manifold of a circle, and have a 1-dimensional vector space at each point on the circle to create an infinitely extended cylinder.

21.7.2 Bundle projection

This is a projection from any point on any of the fibres, to the underlying base manifold.

21.7.3 Trivial and twisted bundles

21.7.4 Cross-sections and zero-sections of fibre bundles

21.7.5 Trivial bundles and the torus

Trivial bundles

The torus

$S_1 \times S_1$

21.7.6 Twisted bundles and the Klein bottle

Twisted bundles

Klein bottles

$S_1 \times S_1$, but twisted

21.7.7 Mobius strips

$S_1 \times$ line segment.

21.8 Other

21.8.1 Submanifolds

Submanifold: subset of manifold which is also manifold

Eg: circle inside a sphere

21.8.2 Boundries and interiors

Around every manifold of dimension n is a boundry of dimension $(n-1)$.

Homeomorphism at boundry: one coordinate always ≥ 0. reduced dimension.

Interior is rest.

21.8.3 Embeddings and immersions

Whitney embedding theroem: all manfiolds can be embedded in R^n space for some n.

21.8.4 Topological groups

We have two operations for groups: multiplication and inversion.

A group is topological if these functions are continuous. + need to just read up on this. where is this relevant? + topological space

For these functions to be continous we need a metric defined on the group.

Chapter 22

Differentiable manifolds

22.1 Introduction

22.1.1 Differentiable transition maps

Transition map recap

Given two charts with an overlap, we have a transition mapping between the two charts of the overlap, where the mapping corresponds to a position on the manifold.

Differentiable transition maps

If this mapping is differentiable, we have a differentiable manifold.

Smooth manifolds

If transition maps are smooth (C^{∞}) then the manifold is smooth.

22.1.2 Differentiable and smooth manifolds

22.1.3 Diffeomorphisms

22.2 Tangent space

22.2.1 Tangent space and tangent vectors

Take a topological space: can all subsets in the toplogy be mapped to n dimensional space? if so, manifold

For this we need openness: a graph for example isn't open and so isn't a manifold

We also need the same number of dimensions at each point

Isn't always the case. eg two circles conneceted by a line is not a manifold. it's 2d in circles, 1d on line (and 3d at connections)

We have a homeomorphism from each point in the toplogy to an n dimensional coordinate system

We also have homeomorphisms of transformation maps, between different points on the topology

The vector space from the homeomorphism is tangent to the manifold at that point. the set of all tangents forms a tangent space

Interior: M; boundry δM Tangent on a manifold:

The tangent space of manifold M at point p is denoted TM_p.

If we have a normal field

$v = v^i e_i$

Then we can differentiate wrt a direction x.

$$\frac{\delta}{\delta x} v = \frac{\delta}{\delta x} v^i e_i$$

$$\frac{\delta}{\delta x} v = e_i \frac{\delta v^i}{\delta x}$$

Because the basis does not change.

If the basis does change we instead have:

$$\frac{\delta}{\delta x} v = \frac{\delta}{\delta x} v^i e_i$$

$$\frac{\delta}{\delta x} v = e_i \frac{\delta v^i}{\delta x} + v^i \frac{\delta e_i}{\delta x}$$

General point. basis can vary across manifold

After this basis diff

Tangent space as vector bundle

Christoffel symbols (page)

Christoffel symbols are connections.

The torsion tensor (own page)

Torsion tensor is

$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$

If torsion is 0, then the connection is symmetric.

Basis of tangent space

We can use as the basis for tangent space:

$\{\frac{(\delta}{\delta x^1})_p, (\frac{\delta}{\delta x^2})_p, ...\}$

This means we can write a tangent vector as:

$u = u^i(\frac{\delta}{\delta x^i})_p$

Basis of contant space

We can use as the basis for the contangent space:

$\{dx^1, dx^2, ...\}$

Metric on the tangent space (to Riemann)

Basis of metric (to Riemann)

The metric depends on the basis too:

$g_{ij}(p) = g((\frac{\delta}{\delta x^i})_p, (\frac{\delta}{\delta x^j})_p)$

The metric on two tangent vectors is defined on the components.

$g = g_{ij}(p)u^i v^j$

22.2.2 Cotangent space and cotangent vectors

22.3 Connections

22.3.1 Transport

22.3.2 Covariant derivative

Essentially as we move across path, we are changing the basis.

We can look at how basis vector change as we translate

We can define as basis as:

$e_i = \frac{\delta x}{\delta x_i}$

How to measure transport

If we take a vector and move it around a curved surface and return it to the same point, it may not face the same way

Eg if you're on equator, move east, north, south to equator, you'll face diffrent direction

This is true on smaller movements of a curved surface

We can use this to measure curvature of a manifold without coordinates

22.3.3 New

covariant derivative. how does change in field compare to parallel transport from curent position?

$\nabla_v(X) = \lim_{t\to 0}\frac{X(p+tv)-X(p)}{t}$

We have point p. We can compare how field in tangent space varies in direction of v.

we don't define basis as each point, but rather how basis changes as you move along a curve

22.3.4 Affine connections

If we have a tangent vector at one point of the manifold, we can map it to a tangent vector at a nearby point on the manifold.

We can use chain rule. so we can have coordinate maps where there is no overlap.

Smooth connections

Affine connection

We have a vector in a tangent space

We have a curve on the manifold from the start point

As we "roll" the tangent, there is a unique vector in each new tangent, determined by transition map

These are affine transformations

Given two points, what path? what transfomration? if curuved then different paths will given different transformation.

22.3.5 Parallel transport

Move and therefore change basis, but components are the same.

22.4 Sort

22.4.1 Orientability of surfaces

Chapter 23

Riemann manifolds

23.1 Introduction

23.1.1 Metric tensors

A metric tensor assigns a bilinear form to each point on the manifold.

We can then take two vectors in the tangent space and return a scalar.

23.1.2 Riemann manifolds and pseudo-Riemann manifolds

Riemann manifolds

Metric is positive definite.

Pseudo-Riemann manifolds

The metric isn't necessarily positive definite.

23.1.3 Metric tensor field

metric tensor field assigns a metric tensor to each point. metric tensor is defined on the tangent bundle. so we have metric on each tangent bundle, but the metric can change thoughout the manifold

23.1.4 Length of paths in Riemann manifolds

We can work out the length of a path through a Riemann manifold.

The geodesic is the shortest such path.

The Riemann metric between two points is the length of the geodesic.

23.2 Connections on Riemann manifolds

23.2.1 Metric compatibility

If we have two vectors in the tangent space of a manifold with a metric tensor, we can get a scalar:

$v^i u^j g_{ij}$

Transported metric

If we transport two vectors along a connection, we have the metric at the new point.

Metric preserving connections

If the connection preserves the metric, then the connection is metric compatible.

23.2.2 Torsion tensor

23.2.3 The Levi-Civita connection

For any metric tensor there is only one connection which preserves the metric and is torsion free.

23.3 Sort

23.3.1 The circle as a topology

23.3.2 Cylinders

23.3.3 Embeddings and immersions

23.3.4 Conformal maps

23.3.5 Geodesics

How do we have straight line on a curve? eg going round equator, but not going via uk.

Take start direction and find tangent vectors. geodesic is where tangent vectors stay parallel.

23.3.6 Curvature tensor

23.3.7 Ricci curvature

Part V

Other topology

Chapter 24

Measure space

24.1 Defining measure spaces

24.1.1 Measure space

In a metric space, the structure was defining a value for each two elements of the set.

In a measure space, the structure defines a value of subsets of the set.

A measure space includes the set X, subsets of the set, Σ, and a function μ which maps from Σ to $\mathbb{R}$.

Sigma algebra

Requirement for Σ.

24.1.2 Axioms for measures

Measures are non-negative

$\forall E \in \Sigma : \mu(E) \geq 0$

The measure for the null set is 0.

$\mu() = 0$

Disjoint sets are additive

$\mu(\vee_{k=1}^{\infty} E_k) = \sum \mu(E_k)$

Where all elements E_k are disjoint. That is, they have no elements in common.

24.2 Examples of measure spaces

24.2.1 The counting measure

$\mu(E)$

This provides the number of elements in E.

24.2.2 The probability measure

This is discussed in more detail in Statistics.

Chapter 25

Graph theory

25.1 Undirected graphs

25.1.1 Vertices and edges

A graph is a set of vertices V, a set of edges E which are subset of pairs from V.

undirected so each edge is a set

Degree of a vertex

The degree of a vertex is the number of edges connections to it.

25.1.2 Order and size of graphs

The order of a graph is the number of vertices, $|V|$.

The size of a graph is the number of edges, $|E|$.

25.1.3 Subgraphs

We can take a subset of vertices, and all edges which only depend on these vertices. This is an induced subgraph.

Induced subgraph

25.1.4 Loops, multiple edges and simple graphs

Loops

A loop is an edge where both the vertices are the same.

Multiple edges

If there are two edges with the same pair of indices, there are multiple edges.

Simple graphs

No loops or multiple edges.

25.2 Directed graphs

25.2.1 Direct acyclic graphs

25.3 Weighted graphs

25.3.1 Edge-weighted graph

An edge-weighted graphs has weights for each edge.

25.3.2 Vertex-weighted graph

A vertex-weighted graph has weights for each vertex.

25.4 Graph representation

25.4.1 Adjacency matrix

We can represent a finite graph as a square matrix. m_{ij} is the number of edges connecting vertex i to vertex j.

25.4.2 Incidence matrix

An incidence matrix has m_{ij} representing the number of connections from vertex i to edge j.

25.4.3 Degree matrix

A degree matrix is a diagonal matrix. Each diagonal contains the degree of the a vertex.

25.4.4 Laplacian matrix

The Laplacian matrix L is formed using the degree matrix D and the adjacency matrix A. $L = D - A$.

25.5 Representing manifolds

25.5.1 Nearest-neighbour graph

25.5.2 Triangular mesh

25.6 Edge colouring

Chapter 26

Category theory

26.1 Category theory

26.1.1 Introduction