## Analytic geometry and Euclidian space

Adam Boult (www.bou.lt)

July 9, 2025

# Contents

| Ι        | Analytic geometry                             | 2  |
|----------|---|----|
| 1        | Points, lines and affine transformations      | 3  |
| <b>2</b> | Euclidian transformations, lengths and angles | 4  |
| 3        | Volumes, perimeters and surface areas         | 8  |
| 4        | 2D polygons                                   | 9  |
| <b>5</b> | 3D polygons                                   | 11 |
| 6        | Algebraic geometry and spheres                | 12 |
| Preface  |   | 13 |

# Part I

Analytic geometry

# Points, lines and affine transformations

- 1.1 Affine spaces
- 1.1.1 Lines
- 1.1.2 Parallel lines

# Euclidian transformations, lengths and angles

### 2.1 Linear metrics

#### 2.1.1 Metrics

We defined a norm as:

 $||v|| = v^T M v$ 

A metric is the distance between two vectors.

 $d(u, v) = ||u - v|| = (u - v)^T M(u - v)$ 

#### Metric space

A set with a metric is a metric space.

#### 2.1.2 Inducing a topology

Metric spaces can be used to induce a topology.

#### 2.1.3 Translation symmetry

The distance between two vectors is:

 $(v-w)^T M(v-w)$ 

So what operations can we do now?

As before, we can do the transformations which preserve  $u^T M v$ , such as the orthogonal group.

But we can also do other translations

 $(v-w)^T M(v-w)$  $v^T M v + w^T M w - v^T M w - w^T M v$ so symmetry is now O(3,1) and affine translations

#### **Translation matrix**

[[1, x][0, 1]] moves vector by x.

### 2.2 Specific groups

- 2.2.1 The affine group
- 2.2.2 The Euclidian group
- 2.2.3 The Galilean group
- 2.2.4 The Poincaré group

### 2.3 Non-linear norms

#### **2.3.1** $L_p$ norms (*p*-norms)

#### $L^P$ norm

This generalises the Euclidian norm.

 $||x||_p = (\sum_{i=1}^n |x|_i^p)^{1/p}$ 

This can defined for different values of p. Note that the absolute value of each element in the vector is used.

Note also that:

 $||x||_2$ 

Is the Euclidian norm.

#### Taxicab norm

This is the  $L^1$  norm. That is:

 $||x||_1 = \sum_{i=1}^n |x|_i$ 

#### Angles

**Cauchy-Schwarz** 

## 2.4 To linear forms

#### 2.4.1 Norms

We can use norms to denote the "length" of a single vector.

$$||v|| = \sqrt{\langle v, v \rangle}$$
$$||v|| = \sqrt{v^* M v}$$

#### Euclidian norm

If M = I we have the Euclidian norm.

$$||v|| = \sqrt{v^* v}$$

If we are using the real field this is:

$$||v|| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

#### Pythagoras' theorem

If n = 2 we have in the real field we have:

$$||v|| = \sqrt{v_1^2 + v_2^2}$$

We call the two inputs x and y, and the length z.

$$z = \sqrt{x^2 + y^2}$$
$$z^2 = x^2 + y^2$$

#### 2.4.2 Angles

#### Recap: Cauchy-Schwarz inequality

This states that:

$$\begin{split} |\langle u,v\rangle|^2 &\leq \langle u,u\rangle \dot{\langle} v,v\rangle \\ \text{Or:} \end{split}$$

 $\langle v,u\rangle \langle u,v\rangle \leq \langle u,u\rangle \dot{\langle} v,v\rangle$ 

#### Introduction

$$\begin{split} \langle v, u \rangle \langle u, v \rangle &\leq \langle u, u \rangle \dot{\langle} v, v \rangle \\ \frac{\langle v, u \rangle \langle u, v \rangle}{||u||.||v||} &\leq ||u||.||v|| \end{split}$$

$$\frac{||u||.||v||}{\langle v, u \rangle} \ge \frac{\langle u, v \rangle}{||u||.||v||}$$
$$\cos(\theta) = \frac{\langle u, v \rangle}{||u||.||v||}$$

## 2.5 Other

2.5.1 Convex hulls

# Volumes, perimeters and surface areas

## 2D polygons

## 4.1 Elementary geometry in 2 dimensions

4.1.1 Triangles

Area of a triangle Circumference of a triangle Sum of angles of a triangle Angles in a triangle add to  $\pi$ .

4.1.2 Quadrilaterals

4.1.3 Oblongs Area of an oblong Circumference of an oblong 4.1.4 Squares Area of a square  $A = l^2$ Circumference of a square C = 4lAngles in a square

Angles in a square sum to  $2\pi$ .

CHAPTER 4. 2D POLYGONS

- 4.1.5 Pentagon
- 4.2 Other
- 4.2.1 Border
- 4.2.2 Interior
- 4.2.3 Open
- 4.2.4 Closed
- 4.2.5 Self-intersecting polygon

# 3D polygons

## 5.1 Elementary geometry in 3 dimensions

- 5.1.1 Pyramid
- 5.1.2 Cubes

Volume of a cube:  $V = l^3$  Surface area of a cube:  $A = 6l^2$ 

# Algebraic geometry and spheres

## 6.1 Circles

6.1.1 Defining circles  $x^2 + y^2 = r^2$ 

**6.1.2** Area of a circle  $A = \pi r^2$ 

6.1.3 Circumference of a circle  $C = 2\pi r$ 

6.2 Spheres

6.2.1 Defining spheres  $x^2 + y^2 + z^2 = r^2$ 

6.2.2 Volume of a sphere *V* =

6.2.3 Surface area of a sphere *A* =

# Preface

This is a live document, and is full of gaps, mistakes, typos etc.