Analytic geometry and non-relativistic field theory

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Analytic geometry

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Points, lines and affine transformations

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2.1.1 Metrics

We defined a norm as:

$||v|| = v^T M v$

A metric is the distance between two vectors.

$d(u, v) = ||u - v|| = (u - v)^T M (u - v)$

Metric space

A set with a metric is a metric space.

2.1.2 Inducing a topology

Metric spaces can be used to induce a topology.

2.1.3 Translation symmetry

The distance between two vectors is:

$(v - w)^T M (v - w)$

So what operations can we do now?

As before, we can do the transformations which preserve $u^T M v$, such as the orthogonal group.

But we can also do other translations

$(v-w)^T M(v-w)$

$v^T Mv + w^T Mw - v^T Mw - w^T Mv$

so symmetry is now $O(3,1)$ and affine translations

Translation matrix

$[[1, x][0, 1]]$ moves vector by x.

2.2 Specific groups

2.2.1 The affine group

2.2.2 The Euclidian group

2.2.3 The Galilean group

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2.3 Non-linear norms

2.3.1 L_p norms (p-norms)

L^P norm

This generalises the Euclidian norm.

$||x||_p = (\sum_{i=1}^n |x|_i^p)^{1/p}$

This can defined for different values of p. Note that the absolute value of each element in the vector is used.

Note also that:

$||x||_2$

Is the Euclidian norm.

Taxicab norm

This is the L^1 norm. That is:

$||x||_1 = \sum_{i=1}^n |x|_i$

Angles

Cauchy-Schwarz

2.4 To linear forms

2.4.1 Norms

We can use norms to denote the "length" of a single vector.

$||v|| = \sqrt{\langle v, v\rangle}$

$||v|| = \sqrt{v^* M v}$

Euclidian norm

If $M = I$ we have the Euclidian norm.

$||v|| = \sqrt{v^* v}$

If we are using the real field this is:

$||v|| = \sqrt{\sum_{i=1}^n v_i^2}$

Pythagoras' theorem

If $n = 2$ we have in the real field we have:

$||v|| = \sqrt{v_1^2 + v_2^2}$

We call the two inputs x and y, and the length z.

$z = \sqrt{x^2 + y^2}$

$z^2 = x^2 + y^2$

2.4.2 Angles

Recap: Cauchy-Schwarz inequality

This states that:

$|\langle u, v\rangle|^2 \leq \langle u, u\rangle \dot{\langle} v, v\rangle$

Or:

$\langle v, u\rangle\langle u, v\rangle \leq \langle u, u\rangle \dot{\langle} v, v\rangle$

Introduction

$\langle v, u\rangle\langle u, v\rangle \leq \langle u, u\rangle \dot{\langle} v, v\rangle$

$\frac{\langle v, u\rangle\langle u, v\rangle}{||u||.||v||} \leq ||u||.||v||$

$$\frac{||u||.||v||}{\langle v,u\rangle} \geq \frac{\langle u,v\rangle}{||u||.||v||}$$

$$\cos(\theta) = \frac{\langle u,v\rangle}{||u||.||v||}$$

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Area of a triangle

Circumference of a triangle

Sum of angles of a triangle

Angles in a triangle add to π.

4.1.2 Quadrilaterals

4.1.3 Oblongs

Area of an oblong

Circumference of an oblong

4.1.4 Squares

Area of a square

$A = l^2$

Circumference of a square

$C = 4l$

Angles in a square

Angles in a square sum to 2π.

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$V = l^3$

Surface area of a cube:

$A = 6l^2$

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6.1 Circles

6.1.1 Defining circles

$x^2 + y^2 = r^2$

6.1.2 Area of a circle

$A = \pi r^2$

6.1.3 Circumference of a circle

$C = 2\pi r$

6.2 Spheres

6.2.1 Defining spheres

$x^2 + y^2 + z^2 = r^2$

6.2.2 Volume of a sphere

$V =$

6.2.3 Surface area of a sphere

$A =$

Part II

Mechanics with a constant field: SUVAT

Chapter 7

Newtonian mechanics

7.1 Introduction

7.1.1 SUVAT

Introduction

For a constant acceleration environment we want to find equations to link:

- Initial speed: v_{t_0}
- End speed: v_{t_1}
- Time: $t_1 - t_0$
- Acceleration: a
- Displacement $s_{t_1} - s_{t_0}$

The SUVAT equations

These are the following, and are derived below.

- $v_{t_1} = a(t_1 - t_0) + v_{t_0}$
- $(s_{t_1} - s_{t_0}) = v_{t_0}(t_1 - t_0) + \frac{1}{2}a(t_1 - t_0)^2$
- $(s_{t_1} - s_{t_0}) = v_{t_1}(t_1 - t_0) - \frac{1}{2}a(t_1 - t_0)^2$
- $v_{t_1}^2 = v_{t_0}^2 + 2a(s_{t_1} - s_{t_0})$
- $(s_{t_1} - s_{t_0}) = (t_1 - t_0)\frac{v_{t_1} + v_{t_1}}{2}$

Equation 1: No displacement

This equation is:

$v_{t_1} = a(t_1 - t_0) + v_{t_0}$

To derive this start with:

$v_t := \frac{\delta s_t}{\delta t}$

$a := \frac{\delta v_t}{\delta t}$

If acceleration is constant, then

$\frac{\delta v_t}{\delta t} = a$

$v_t = \int a dt + v_0$

$v_t = at + v_0$

Equation 2: No end velocity

This equation is:

$(s_{t_1} - s_{t_0}) = v_{t_0}(t_1 - t_0) + \frac{1}{2}a(t_1 - t_0)^2$

To derive this start with:

$v := \frac{\delta s_t}{\delta t}$

Then:

$\frac{\delta s_t}{\delta t} = at + v_0$

$s_t = \frac{1}{2}at^2 + v_0 t + s_0$

$(s_t - s_0) = v_0 t + \frac{1}{2}at^2$

Equation 3: No start velocity

This equation is:

$(s_{t_1} - s_{t_0}) = v_{t_1}(t_1 - t_0) - \frac{1}{2}a(t_1 - t_0)^2$

To derive this start with:

$v_t = at + v_0$

$(s_t - s_0) = t\frac{v_t + v_0}{2}$

So:

$v_0 = v_t - at$

$v_0 = \frac{2}{t}(s_t - s_0) - v_t$

$v_t - at = \frac{2}{t}(s_t - s_0) - v_t$

$(s_t - s_0) = v_t t - \frac{1}{2}at^2$

Equation 4: No time

This equation is:

$v_{t_1}^2 = v_{t_0}^2 + 2a(s_{t_1} - s_{t_0})$

To derive this start with:

$v_t = at + v_0$

$(s_t - s_0) = t\frac{v_t + v_0}{2}$

So:

$t = \frac{v_t - v_0}{a}$

$t = 2\frac{s_t - s_0}{v_t + v_0}$

$\frac{v_t - v_0}{a} = 2\frac{s_t - s_0}{v_t + v_0}$

$(v_t - v_0)(v_t + v_0) = 2a(s_t - s_0)$

$v_t^2 = v_0^2 + 2a(s_t - s_0)$

Equation 5: No acceleration

This equation is:

$(s_{t_1} - s_{t_0}) = (t_1 - t_0)\frac{v_{t_1} + v_{t_1}}{2}$

To derive this start with:

$v_t = at + v_0$

$s_t - s_0 = \frac{1}{2}at^2 + v_0 t$

So:

$a = \frac{v_t - v_0}{t}$

$a = \frac{2[(s_t - s_0) - v_0 t]}{t^2}$

$$\frac{v_t - v_0}{t} = \frac{2[(s_t - s_0) - v_0 t]}{t^2}$$

$$t(v_t - v_0) = 2[(s_t - s_0) - v_0 t]$$

$$t(v_t + v_0) = 2(s_t - s_0)$$

$$(s_t - s_0) = t\frac{v_t + v_0}{2}$$

Part III

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Harmonics

8.1 Introduction

8.1.1 Introduction

Acceleration inversely proportional to distance.

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9.1 Introduction

9.1.1 Deriving Galileo's laws from Newton

9.1.2 Deriving Copernicus's laws from Newton

Chapter 10

Music

10.1 Introduction

10.1.1 Pure tones

Describe a pitch by vibration frequency Hertz Hz, period

octave up is 2x frequency

phenomenon of finding similarity betwenn octaves, so can be named similarly.

Musical interval 2:1 - octave 1:1 - unison

consonance and dissonance phenomena of whether we see pitches as in harmony or not

how to create notes between? equal temperament: want [n] number, so each ratio is $2^{1/n}$

for 12 tone scale, $2^{1/12}$

harmonic of a tone take tone. integer multiples of the frequencies are the harmonic sequence. frequencies here are "rational" frequences of equal temperament are irrational (between octaves)

pythagorean scale generated by pure 5ths (3:2)(aka perfect fifth) and octaves (2:1) result is unevenly spaced notes

10.1.2 Chords

major chord: root, major third, perfect fifth minor chord: root, minor third, perfect fifth

Ohm's acoustic law: we can hear individual notes when put together

Part IV

2 body gravity

Part V

3 body gravity

Part VI

Dynamical systems: Attractors and strange attractors

Part VII

Geometrical optics

Part VIII

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Part IX

Physical models

Chapter 11

Paths

11.1 Describing paths

11.1.1 Describing events

In vector space $\mathbb{R}^n$.

$\mathbf{q} \in \mathbb{R}^n$

11.1.2 Describing the path of a particle

Also known as a worldline.

Index to t

$\mathbf{q}(t)$

11.1.3 Describing the velocity of a particle

$v = \frac{\delta \mathbf{q}}{\delta t}$

11.1.4 Describing the acceleration of a particle

$a = \frac{\delta v}{\delta t}$

$a = \frac{\delta^2 \mathbf{q}}{\delta t^2}$

11.2 Action

11.2.1 Action

We observe a particle moving in a path. We want to model the path that the particle takes.

The path is in a vector space, with coordinates $\mathbf{q}$. These coordinates could refer to the x, y, z and t coordinates we are are familar with.

For the path we have a start point a and end point b. We can define the length of the path as:

$S = \int_a^b d\tau$

We call S the action.

11.2.2 Linear metrics

We use a linear metric.

$\tau^2 = \mathbf{q}^T\mathbf{M}\mathbf{q}$

So:

$d\tau^2 = (d\mathbf{q})^T\mathbf{M}d\mathbf{q}$

$S = \int_a^b \sqrt{(d\mathbf{q})^T\mathbf{M}d\mathbf{q}}$

11.2.3 Time and velocity

$S = \int_a^b \sqrt{\frac{1}{dt^2}(d\mathbf{q})^T\mathbf{M}d\mathbf{q}}dt$

$S = \int_a^b \sqrt{(\frac{d\mathbf{q}}{dt})^T\mathbf{M}\frac{d\mathbf{q}}{dt}}dt$

$S = \int_a^b \sqrt{(\dot{\mathbf{q}})^T\mathbf{M}\dot{\mathbf{q}}}dt$

11.3 The Lagrangian

11.3.1 Lagrangians

We have:

$S = \int_a^b \sqrt{(\dot{\mathbf{q}})^T\mathbf{M}\dot{\mathbf{q}}}dt$

We can define:

$L = \sqrt{(\dot{\mathbf{q}})^T\mathbf{M}\dot{\mathbf{q}}}$

So we have:

$S = \int_a^b L dt$

11.3.2 Principle of stationary action

$\delta A = 0$

That is, the coordinates and their velocities are such that action is stationary.

11.3.3 Euler-Lagrange

We have $q(t)$ which makes the action stationary. Consider adding proportion ϵ of another function $f(t)$ to $q(t)$.

$A' = \int_{t_0}^{t_1} L[q(t) + \epsilon f(t), \dot{q(t)} + \epsilon f'(t)]dt$

$\frac{A' - A}{\epsilon} = \frac{1}{\epsilon} \int_{t_0}^{t_1} L[q(t) + \epsilon f(t), \dot{q(t)} + \epsilon f'(t)] - L[q, \dot{q}]dt$

We can do a Taylor expansion of A'.

$A' = \int_{t_0}^{t_1} L[q(t) + \epsilon f(t), \dot{q(t)} + \epsilon f'(t)]dt$

$A' = \int_{t_0}^{t_1} L[q(t), \dot{q(t)}] + \epsilon[f \frac{\delta L}{\delta q} + f \cdot \frac{\delta L}{\delta \dot{q}}] + \epsilon^2[...]dt$

So:

$\frac{A' - A}{\epsilon} = \frac{1}{\epsilon} \int_{t_0}^{t_1} L[q(t), \dot{q(t)}] + \epsilon[f \frac{\delta L}{\delta q} + f \cdot \frac{\delta L}{\delta q \cdot}] + \epsilon^2[...] - L[q, \dot{q}]dt$

$\frac{A' - A}{\epsilon} = \int_{t_0}^{t_1} [f \frac{\delta L}{\delta q} + f \cdot \frac{\delta L}{\delta \dot{q}}] + \epsilon[...]dt$

We can now make the left side 0, by using the definition of stationary action.

$\lim_{\epsilon \to 0} \frac{A' - A}{\epsilon} = \int_{t_0}^{t_1} [f \frac{\delta L}{\delta q} + f \cdot \frac{\delta L}{\delta \dot{q}}]dt$

$\int_{t_0}^{t_1} [f \frac{\delta L}{\delta q} + f \cdot \frac{\delta L}{\delta \dot{q}}]dt = 0$

$\int_{t_0}^{t_1} [f \frac{\delta L}{\delta q}]dt + \int_{t_0}^{t_1} [f \cdot \frac{\delta L}{\delta \dot{q}}]dt = 0$

Note that

$\int_{t_0}^{t_1} [f \cdot \frac{\delta L}{\delta \dot{q}}]dt = [f \frac{\delta L}{\delta \dot{q}}]_{t_0}^{t_1} - \int_{t_0}^{t_1} f \frac{d}{dt} \frac{\delta L}{\delta \dot{q}} dt$

We assume that $f(t_0) = f(t_1) = 0$ and so:

$\int_{t_0}^{t_1} [f \cdot \frac{\delta L}{\delta \dot{q}}]dt = - \int_{t_0}^{t_1} f \frac{d}{dt} \frac{\delta L}{\delta \dot{q}} dt$

Plugging this back in we get:

$\int_{t_0}^{t_1} [f \frac{\delta L}{\delta q}] - f \frac{d}{dt} \frac{\delta L}{\delta \dot{q}} dt] = 0$

$\int_{t_0}^{t_1} f[\frac{\delta L}{\delta q}] - \frac{d}{dt} \frac{\delta L}{\delta \dot{q}} dt] = 0$

Since this applies to all possible functions we get:

$\frac{\delta L}{\delta q} = \frac{d}{dt} \frac{\delta L}{\delta \dot{q}}$

11.3.4 Definition: Momentum

$p = \frac{\delta L}{\delta \dot{q}}$

11.3.5 EL v2

$L = \sqrt{(\dot{\mathbf{q}})^T \mathbf{M} \dot{\mathbf{q}}}$

$\frac{\delta L}{\delta q} - \frac{d}{dt} (\frac{\delta L}{\delta \dot{q}}) = 0$

We have

$S = \int_a^b L(q(t), \dot{q}(t)) dt$

$\delta S = \delta \int_a^b L(q(t), \dot{q}(t)) dt$

$J = \int_a^b L(t, q(t), \dot{q}(t)) dt$

$J = \sum_{k=0}^{n-1}$

11.3.6 EL v3

$A = \sum L(x(t), \dot{x}(t)) \delta t$

$A = \sum L(\frac{x(t) + x(t-1)}{2}, \frac{x(t) - x(t-1)}{\delta t}) \delta t$

$\frac{\delta}{\delta x(t)} A = \sum \frac{\delta}{\delta x(t)} L(\frac{x(t) + x(t-1)}{2}, \frac{x(t) - x(t-1)}{\delta t}) \delta t$

$\frac{\delta}{\delta x(t)} A = \delta t [\frac{\delta}{\delta x(t)} L(\frac{x(t) + x(t-1)}{2}, \frac{x(t) - x(t-1)}{\delta t}) + \frac{\delta}{\delta x(t)} L(\frac{x(t+1) + x(t)}{2}, \frac{x(t+1) - x(t)}{\delta t})]$

$\frac{\delta}{\delta x(t)} A = \delta t [\frac{1}{2} L_x + \frac{1}{\delta t} L_{\dot{x}} + \frac{1}{2} L_x - \frac{1}{\delta t} L_{\dot{x}}]$

$A = \int_a^b L(q(t), \dot{q}(t)) dt$

$A = \sum L(q(t), \dot{q}(t)) \delta t$

$A = \sum L(\frac{q(t) + q(t-1)}{2}, \frac{q(t) - q(t-1)}{\delta t}) \delta t$

$$\frac{\delta}{\delta q_i(t)}A = \sum \frac{\delta}{\delta q_i(t)}L(\frac{q(t)-q(t-1)}{2}, \frac{q(t)-q(t-1)}{\delta t})\delta t$$

$$\frac{\delta}{\delta q_i(t)}A = \delta t[\frac{\delta}{\delta q_i(t)}L(\frac{q(t)+q(t-1)}{2}, \frac{q(t)-q(t-1)}{\delta t})+\frac{\delta}{\delta q_i(t)}L(\frac{q(t+1)+q(t)}{2}, \frac{q(t+1)-q(t)}{\delta t})]$$

$$\frac{\delta}{\delta q_i(t)}A = \delta t[\frac{1}{2}L_{q_i}+\frac{1}{\delta t}L_{\dot{q}_i}+\frac{1}{2}L_{q_i}-\frac{1}{\delta t}L_{\dot{q}_i}+\frac{\delta}{\delta q_i(t)}L(\frac{q(t+1)+q(t)}{2}, \frac{q(t+1)-q(t)}{\delta t})]$$

11.4 The Euclidian metric

11.4.1 The Euclidian metric

For the Euclidian metric:

$M = I$

$(dv)^T M dv = (dv)^T dv = dx^2 + dy^2 + dz^2$

$Action = \int \sqrt{dx^2 + dy^2 + dz^2}$

$Action = \int \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt$

$Action = \int v dt$

What are symmetries here? Gallilean group and?

11.4.2 Euclidian rotations

11.4.3 The Euclidan group

11.4.4 The Galilean group

11.5 Examples from Euler-Lagrange

11.5.1 Outcome

$\frac{\delta L}{\delta q} = \frac{d}{dt}\frac{\delta L}{\delta \dot{q}}$

$L = \sqrt{(\dot{\mathbf{q}})^T \mathbf{M} \dot{\mathbf{q}}}$

If $M = I$, then:

$L = \sqrt{(\dot{\mathbf{q}})^T \dot{\mathbf{q}}}$

In 1 dimension, Euclid:

$L = \sqrt{\dot{q}^2}$ $L = \dot{q}$

So:

$\frac{\delta L}{\delta q} = \frac{\delta}{\delta q}\dot{q}$ $\frac{\delta L}{\delta q} = 0$

$\frac{\delta L}{\delta \dot{q}} = \frac{\delta}{\delta \dot{q}}\dot{q}\ \frac{\delta L}{\delta \dot{q}} = 1$

Into Euler-Lagrange:

$\frac{\delta L}{\delta q} = \frac{d}{dt}\frac{\delta L}{\delta \dot{q}}\ 0 = \frac{d}{dt}1$

In three dimensions:

$L = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

So:

$\frac{\delta L}{\delta q} = \frac{\delta}{\delta q}\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\ \frac{\delta L}{\delta q} = 0$

$\frac{\delta L}{\delta \dot{q}} = \frac{\delta}{\delta \dot{q}}\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

11.6 Other

11.6.1 Momentum

We define the momentum as:

$p_j = \frac{\delta L}{\delta \dot{q}_j}$

$p_j = \frac{\delta}{\delta \dot{q}_j}\sqrt{(\dot{\mathbf{q}})^T \mathbf{M}\dot{\mathbf{q}}}$

11.6.2 Force

Chapter 12

Symmetry

12.1 Introduction

12.1.1 Symmetry

We have a system of particles or something. we can do a measure on it.

We can do functions on the system.

What is preserved is an invariant measure

We observe event, worldline. other observers can:

12.1.2 Rotations

12.1.3 Translations

12.1.4 Boosts

Part here on adding velocities. show classical limit of normal adding them.

Chapter 13

Fields

13.1 Introduction

13.1.1 Fields

13.1.2 Action on a field

13.1.3 The Euler-Lagrange equations for fields

Part X

Electromagnetism

Chapter 14

Electromagnetism, waves, action on a field, least action on a field and gauge theory

14.1 Introduction

14.1.1 Introduction

$\overrightarrow{\mathbf{F}} = e\overrightarrow{\mathbf{E}}$ $\overrightarrow{\mathbf{F}} = e\overrightarrow{v} \times \overrightarrow{\mathbf{B}}$ $\overrightarrow{\mathbf{F}} = e\overrightarrow{\mathbf{E}} + \overrightarrow{v} \times \overrightarrow{\mathbf{B}}$
$div\overrightarrow{\mathbf{B}} = 0$ $div\overrightarrow{\mathbf{E}} =$

14.1.2 Electric fields

14.1.3 Magnetic fields

14.1.4 Electric potential

14.1.5 Magnetic potential

14.1.6 Column's law

$\overrightarrow{\mathbf{F}} = k_e \frac{q_1 q_2}{r^2}$

Part XI

Lagrangian formations

Part XII

Hamiltonian formations

Chapter 15

Hamiltonian

15.1 Introduction

15.1.1 Legendre transformation