Topology

Adam Boult (www.bou.lt)

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1.1 Introduction

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Chapter 2

Topology of finite sets

2.1 Nearness functions

2.1.1 Topologies

2.1.2 Topologies on sets

T is a topology on set X if:

- $X \in T$
- $\varnothing \in T$
- Unions of T are in T
- Intersections of T are in T

2.1.3 Examples of topologies: The trivial topology

The trivial topology contains only the underlying set and the empty set.

2.1.4 Examples of topologies: The discrete toplogy

The discrete toplogy contains all subsets of the underlying set (is this the power set?)

2.2 Neighbourhoods

2.2.1 Neighbourhood topology

We have a set X.

For each element $x \in X$, there is a non-empty set of neighbourhoods $N \in \mathbf{N}(x)$ where $x \in N \subseteq X$ such that:

- If N is a subset of M, M is a neighbourhood.
- The intersection of two neighbourhoods of x is a neighbourhood of x.
- N is a neighbourhood for each point in some $M \subseteq N$

2.2.2 Topological distinguishability

If two points have the same neighbourhoods then they are topologically indistinguishable.

For example in the trivial topology, all points are topologically indistinguishable.

2.2.3 Open sets

U is an open set if it is a neighbourhood for all its points.

2.3 Open and closed sets

2.3.1 Limit points and closure

Limit points

A point x in the topological set X is a limit point for $S \subset X$ if every neighbourhood of x contains another point in S.

For example -1 is a limit point for the real numbers where S is $[0, 1]$ (or $(0, 1)$.

Closure

The closure of a subset of a topological space is the subset itself along with all limit points.

So the closure of $|x| < 1$ includes -1 and 1.

2.3.2 Boundries and interiors

The boundry of the subset S of a topology is the intersection with the closure of S with the closure of the complement of S.

So the boundry of both $(0, 1)$ and $[0, 1]$ are 0 and 1.

The interior of S is S without the boundry.

So the interior of $(0, 1)$ and $[0, 1]$ are both $(0, 1)$.

2.3.3 Closed sets

The complement of any open set is a closed set.

A set can be open, closed, both or neither.

2.4 Compactness

2.4.1 Covers

A space X is covered by a set of subsets of X, C, if the union of C is X.

2.4.2 Subcover

A subset of C which still covers X is a a subcover.

2.4.3 Open cover

C is an open cover if each member is an open set.

2.4.4 Universal cover

2.4.5 Bases of topologies

Subset B of topology T is a base for T if all elements of T are unions of members of B.

Second-countable space

If B is finite then the toplogy is a second-countable space.

2.5 Separation

2.5.1 Connected and separated sets

Two subsets of X in topological space T are separated if each subset is disjoint from the other's closure.

So $[-1, 0)$ and $(0, 1)$ are separated.

$[-1, 0]$ and $(0, 1)$ are not separated.

Sets which are not separated are connected.

2.6 Cartesian products

2.6.1 Box topology

2.6.2 Product topology

2.7 Creating topologies from sets

2.7.1 The trivial topology

A topology which contains just X and $\varnothing$ is the trivial topology.

2.7.2 Discrete topology

2.8 Taxonomy of spaces

2.8.1 Lindelöf space

In a Lindelöf space all open covers have countable subcovers.

This is weaker than compactness, which requires that every open cover has a finite subcover.

2.8.2 Kolmogorov space

In a Kolmogorov (or T_0) space, for every pair of points there is a neighbourhood containing one but not the other.

2.9 Local properties

2.9.1 Local properties

Locally, a topology may have properties which are not present globally.

2.9.2 Locally compact spaces

2.9.3 Locally connected spaces

2.10 TO INF

2.10.1 Hausdorf space

In a Hausdorf (or T_2) space, any two different points have neighbourhoods which are disjoint.

2.10.2 Compact spaces

A space X is compact if each open cover has a finite subcover.

If we can define a cover which does not have a finite subcover, then the space is not compact.

For example an infinite cover could be tend towards $(0, 1)$, eg as $\frac{1}{n}, 1 - \frac{1}{n}$

This covers $(0, 1)$, but there is no finite subcover. As a result $(0, 1)$ is not compact.

Chapter 3

Topological manifolds

3.1 Introduction

3.1.1 Manifolds, charts and atlases

A manifold is a set of points and associated charts.

A chart is a mapping from each point in a subset of the manifold to a point in a vector space.

These charts are invertible. If we are given coordinates, we can identify the point in the manifold it comes from.

For each point we have a topological neighbourhood. For each point in the neighbourhood, we can map to an element in the tangent space.

Example: The sphere

We can map a hemisphere to a subset of R^2. Given a point in R^2 we can identify a specific point on the hemisphere, and given a s specific point on the hemisphere we can identify a point in R^2.

Universal charts

If the vector space is flat and non-repeating, then a single chart can be used to map the whole manifold.

Atlases

If we have a collection of charts which covers each point needs to be covered at least once, we have an atlas. Each chart needs to be to the same dimensional vector space.

3.1.2 Transition maps

Where two charts overlap we can express the points where the charts overlap as two different coordinates.

We can express the mapping from these coordinates as a function. This is a transition map.

Overlapping charts

If two charts cover some of the same points on a manifold then we can define a function for those points where we move from one vector to another.

We can represent moving between charts as:

ab^{-1}

3.1.3 Mapping 2D manifolds to Riemann surfaces

Needs to be orientable and metricisable.

3.1.4 Connections of topological manifolds

Connected vs path-connected topological manifods.

3.2 Dimension theory

3.2.1 Refinement

3.2.2 Ply (order) of a cover

3.2.3 Small inductive dimension

3.2.4 Large inductive dimension

3.2.5 Lebesgue covering dimension

3.3 Paths

3.3.1 Paths and loops

Paths

We have the set X. We define a mapping $[0, 1] \to X$

If a path exists between any two points, then the space is path-connected.

Loops

This is a path which ends on itself.

If $f(0) = f(1)$ then it is a loop.

3.3.2 Holes and genuses

Holes

Genes

The genus of a topology is the number of holes in the topology.

3.3.3 Path-connect spaces

3.4 Simply-connected 2D manifolds

3.4.1 Elliptic (Riemann sphere)

3.4.2 Parabolic (complex plane)

3.4.3 Hyperbolic (open disk)

3.5 Not simply-connected 2D manifolds

3.5.1 Torus

3.5.2 Hyper-elliptic curves

3.6 Functions between topologies

3.6.1 Functions between topologies

We can define a function from topology to another.

$f(X) = Y$

Continuous functions between topologies

If $f(X)$ is continous, then we have a continous function between topologies.

Inverse functions between topologies

If $f(X)$ is invertible then there is a inverse mapping.

3.6.2 Homotopy

3.6.3 Homeomorphisms

If there is a mapping which is invertible and continuous, it is a homeomorphism.

3.7 Fibre bundles

3.7.1 Vector bundles

A vector bundle consists of a base manifold (a base space), and a real vector space at each point in the base manifold.

Example

For example we can have a base manifold of a circle, and have a 1-dimensional vector space at each point on the circle to create an infinitely extended cylinder.

3.7.2 Bundle projection

This is a projection from any point on any of the fibres, to the underlying base manifold.

3.7.3 Trivial and twisted bundles

3.7.4 Cross-sections and zero-sections of fibre bundles

3.7.5 Trivial bundles and the torus

Trivial bundles

The torus

$S_1 \times S_1$

3.7.6 Twisted bundles and the Klein bottle

Twisted bundles

Klein bottles

$S_1 \times S_1$, but twisted

3.7.7 Mobius strips

$S_1\times$ line segment.

3.8 Other

3.8.1 Submanifolds

Submanifold: subset of manifold which is also manifold

Eg: circle inside a sphere

3.8.2 Boundries and interiors

Around every manifold of dimension n is a boundry of dimension $(n-1)$.

Homeomorphism at boundry: one coordinate always ≥ 0. reduced dimension.

Interior is rest.

3.8.3 Embeddings and immersions

Whitney embedding theroem: all manfiolds can be embedded in R^n space for some n.

3.8.4 Topological groups

We have two operations for groups: multiplication and inversion.

A group is topological if these functions are continuous. + need to just read up on this. where is this relevant? + topological space

For these functions to be continous we need a metric defined on the group.

Chapter 4

Differentiable manifolds

4.1 Introduction

4.1.1 Differentiable transition maps

Transition map recap

Given two charts with an overlap, we have a transition mapping between the two charts of the overlap, where the mapping corresponds to a position on the manifold.

Differentiable transition maps

If this mapping is differentiable, we have a differentiable manifold.

Smooth manifolds

If transition maps are smooth (C^∞) then the manifold is smooth.

4.1.2 Differentiable and smooth manifolds

4.1.3 Diffeomorphisms

4.2 Tangent space

4.2.1 Tangent space and tangent vectors

Take a topological space: can all subsets in the toplogy be mapped to n dimensional space? if so, manifold

For this we need openness: a graph for example isn't open and so isn't a manifold

We also need the same number of dimensions at each point

Isn't always the case. eg two circles conneceted by a line is not a manifold. it's 2d in circles, 1d on line (and 3d at connections)

We have a homeomorphism from each point in the toplogy to an n dimensional coordinate system

We also have homeomorphisms of transformation maps, between different points on the topology

The vector space from the homeomorphism is tangent to the manifold at that point. the set of all tangents forms a tangent space

Interior: M; boundry δM Tangent on a manifold:

The tangent space of manifold M at point p is denoted TM_p.

If we have a normal field

$v = v^i e_i$

Then we can differentiate wrt a direction x.

$\frac{\delta}{\delta x} v = \frac{\delta}{\delta x} v^i e_i$

$\frac{\delta}{\delta x} v = e_i \frac{\delta v^i}{\delta x}$

Because the basis does not change.

If the basis does change we instead have:

$\frac{\delta}{\delta x} v = \frac{\delta}{\delta x} v^i e_i$

$\frac{\delta}{\delta x} v = e_i \frac{\delta v^i}{\delta x} + v^i \frac{\delta e_i}{\delta x}$

General point. basis can vary across manifold

After this basis diff

Tangent space as vector bundle

Christoffel symbols (page)

Christoffel symbols are connections.

The torsion tensor (own page)

Torsion tensor is

$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$

If torsion is 0, then the connection is symmetric.

Basis of tangent space

We can use as the basis for tangent space:

$\{\frac{(\delta}{\delta x^1})_p, (\frac{\delta}{\delta x^2})_p, ...\}$

This means we can write a tangent vector as:

$u = u^i(\frac{\delta}{\delta x^i})_p$

Basis of contant space

We can use as the basis for the contangent space:

$\{dx^1, dx^2, ...\}$

Metric on the tangent space (to Riemann)

Basis of metric (to Riemann)

The metric depends on the basis too:

$g_{ij}(p) = g((\frac{\delta}{\delta x^i})_p, (\frac{\delta}{\delta x^j})_p)$

The metric on two tangent vectors is defined on the components.

$g = g_{ij}(p)u^i v^j$

4.2.2 Cotangent space and cotangent vectors

4.3 Connections

4.3.1 Transport

4.3.2 Covariant derivative

Essentially as we move across path, we are changing the basis.

We can look at how basis vector change as we translate

We can define as basis as:

$e_i = \frac{\delta x}{\delta x_i}$

How to measure transport

If we take a vector and move it around a curved surface and return it to the same point, it may not face the same way

Eg if you're on equator, move east, north, south to equator, you'll face diffrent direction

This is true on smaller movements of a curved surface

We can use this to measure curvature of a manifold without coordinates

4.3.3 New

covariant derivative. how does change in field compare to parallel transport from curent position?

$\nabla_v(X) = \lim_{t\to 0} \dfrac{X(p+tv)-X(p)}{t}$

We have point p. We can compare how field in tangent space varies in direction of v.

we don't define basis as each point, but rather how basis changes as you move along a curve

4.3.4 Affine connections

If we have a tangent vector at one point of the manifold, we can map it to a tangent vector at a nearby point on the manifold.

We can use chain rule. so we can have coordinate maps where there is no overlap.

Smooth connections

Affine connection

We have a vector in a tangent space

We have a curve on the manifold from the start point

As we "roll" the tangent, there is a unique vector in each new tangent, determined by transition map

These are affine transformations

Given two points, what path? what transfomration? if curuved then different paths will given different transformation.

4.3.5 Parallel transport

Move and therefore change basis, but components are the same.

4.4 Sort

4.4.1 Orientability of surfaces

Chapter 5

Riemann manifolds

5.1 Introduction

5.1.1 Metric tensors

A metric tensor assigns a bilinear form to each point on the manifold.

We can then take two vectors in the tangent space and return a scalar.

5.1.2 Riemann manifolds and pseudo-Riemann manifolds

Riemann manifolds

Metric is positive definite.

Pseudo-Riemann manifolds

The metric isn't necessarily positive definite.

5.1.3 Metric tensor field

metric tensor field assigns a metric tensor to each point. metric tensor is defined on the tangent bundle. so we have metric on each tangent bundle, but the metric can change thoughout the manifold

5.1.4 Length of paths in Riemann manifolds

We can work out the length of a path through a Riemann manifold.

The geodesic is the shortest such path.

The Riemann metric between two points is the length of the geodesic.

5.2 Connections on Riemann manifolds

5.2.1 Metric compatibility

If we have two vectors in the tangent space of a manifold with a metric tensor, we can get a scalar:

$v^i u^j g_{ij}$

Transported metric

If we transport two vectors along a connection, we have the metric at the new point.

Metric preserving connections

If the connection preserves the metric, then the connection is metric compatible.

5.2.2 Torsion tensor

5.2.3 The Levi-Civita connection

For any metric tensor there is only one connection which preserves the metric and is torsion free.

5.3 Sort

5.3.1 The circle as a topology

5.3.2 Cylinders

5.3.3 Embeddings and immersions

5.3.4 Conformal maps

5.3.5 Geodesics

How do we have straight line on a curve? eg going round equator, but not going via uk.

Take start direction and find tangent vectors. geodesic is where tangent vectors stay parallel.

5.3.6 Curvature tensor

5.3.7 Ricci curvature

Part III

Other topology

Chapter 6

Measure space

6.1 Defining measure spaces

6.1.1 Measure space

In a metric space, the structure was defining a value for each two elements of the set.

In a measure space, the structure defines a value of subsets of the set.

A measure space includes the set X, subsets of the set, Σ, and a function μ which maps from Σ to $\mathbb{R}$.

Sigma algebra

Requirement for Σ.

6.1.2 Axioms for measures

Measures are non-negative

$\forall E \in \Sigma : \mu(E) \geq 0$

The measure for the null set is 0.

$\mu() = 0$

Disjoint sets are additive

$\mu(\vee_{k=1}^{\infty} E_k) = \sum \mu(E_k)$

Where all elements E_k are disjoint. That is, they have no elements in common.

6.2 Examples of measure spaces

6.2.1 The counting measure

$\mu(E)$

This provides the number of elements in E.

6.2.2 The probability measure

This is discussed in more detail in Statistics.

Chapter 7

Graph theory

7.1 Undirected graphs

7.1.1 Vertices and edges

A graph is a set of vertices V, a set of edges E which are subset of pairs from V.

undirected so each edge is a set

Degree of a vertex

The degree of a vertex is the number of edges connections to it.

7.1.2 Order and size of graphs

The order of a graph is the number of vertices, $|V|$.

The size of a graph is the number of edges, $|E|$.

7.1.3 Subgraphs

We can take a subset of vertices, and all edges which only depend on these vertices. This is an induced subgraph.

Induced subgraph

7.1.4 Loops, multiple edges and simple graphs

Loops

A loop is an edge where both the vertices are the same.

Multiple edges

If there are two edges with the same pair of indices, there are multiple edges.

Simple graphs

No loops or multiple edges.

7.2 Directed graphs

7.2.1 Direct acyclic graphs

7.3 Weighted graphs

7.3.1 Edge-weighted graph

An edge-weighted graphs has weights for each edge.

7.3.2 Vertex-weighted graph

A vertex-weighted graph has weights for each vertex.

7.4 Graph representation

7.4.1 Adjacency matrix

We can represent a finite graph as a square matrix. m_{ij} is the number of edges connecting vertex i to vertex j.

7.4.2 Incidence matrix

An incidence matrix has m_{ij} representing the number of connections from vertex i to edge j.

7.4.3 Degree matrix

A degree matrix is a diagonal matrix. Each diagonal contains the degree of the a vertex.

7.4.4 Laplacian matrix

The Laplacian matrix L is formed using the degree matrix D and the adjacency matrix A. $L = D - A$.

7.5 Representing manifolds

7.5.1 Nearest-neighbour graph

7.5.2 Triangular mesh

7.6 Edge colouring

Chapter 8

Category theory

8.1 Category theory

8.1.1 Introduction

Chapter 9

Projective geometry

9.1 Introduction

9.1.1 Introduction